Covariant four-dimensional scattering equations for the $NN - \pi NN$ system

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Abstract

We derive a set of coupled four-dimensional integral equations for the $NN - \pi NN$ system using our modified version of the Taylor method of classification-of-diagrams. These equations are covariant, obey two and three-body unitarity and contain subtraction terms which eliminate the double-counting present in some previous four-dimensional $NN - \pi NN$ equations. The equations are then recast into a from convenient for computation by grouping the subtraction terms together and obtaining a set of two-fragment scattering equations for the amplitudes of interest.

I. INTRODUCTION

The $NN - \pi NN$ problem occupies a privileged place in nuclear physics. Not only does its history stretch back to Yukawa's original efforts to model the strong nuclear force [1], but it is still the subject of considerable research to this day. This current interest occurs partly because the $NN - \pi NN$ system is one light nuclear system which can be used to test the validity of models in which hadrons are the fundamental degrees of freedom. Only in systems which, like the $NN - \pi NN$ system, contain a relatively small number of degrees of freedom, is it possible to complete a calculation of experimentally observed quantities within the framework of such a model, while retaining some control over the approximations used. Consequently, considerable effort has been put into attempting to produce physically accurate hadronic models of the $NN - \pi NN$ system. The predictions of these models can then be compared with the mass of available experimental data and a judgement on the models' validity formed.

Over the past twenty years considerable theoretical progress has been made in this direction, with the culmination being the independent derivation, by a number of groups using different techniques, of a set of scattering equations known as the $NN - \pi NN$ equations [2–12]. The problem is that models based on these $NN - \pi NN$ equations which treat the $\pi - N$ amplitude as the sum of a pole and non-pole term fail to reproduce the experimental data. This may well be because of certain theoretical inconsistencies in the equations. For example, Jennings [13] has pointed out that the $NN - \pi NN$ equations include the diagram on the right of Figure 1, but exclude that on the left—even though the left-hand diagram is merely a different time-order of the right-hand one and both diagrams represent the same physical process. Jennings and Rinat [13,14] and Mizutani et al. [15]

have made it plausible that the omission of this diagram accounts for the failure of models based on the $NN-\pi NN$ equations to correctly predict the tensor polarization, T_{20} for $\pi - d$ scattering. (For more detail on the history of the theory of the $NN - \pi NN$ system and a thorough comparison with the experimental data, see the recent book by Garcilazo and Mizutani [16].)

The standard $NN-\pi NN$ equations do not include the Jennings mechanism because they are derived by using unitarity as a criterion for truncating the full field theory of nucleons and pions. One way in which this derivation can be done is to examine the diagrammatic expansion for the $NN - \pi NN$ amplitudes in old-fashioned or time-ordered perturbation theory (TOPT). The $NN - \pi NN$ equations may then be obtained by truncating this expansion at the one explicit pion level. (See [11,17] for details.) This truncation not only appears to lead to incorrect predictions for T_{20}, but also produces the under-dressing of the two-nucleon propagator discussed in [18,19]. Indeed, the fundamental shortcoming of the standard $NN - \pi NN$ equations is that two diagrams each representing the same set of physical processes, but having them occur in a different time-order may end up treated completely differently. In fact, as we see in Fig. 1, one may be included and one excluded, with potentially disastrous consequences.

This difficulty can be entirely circumvented, however, if the diagrammatic expansion of the field theory of nucleons and pions is performed in covariant perturbation theory, rather than time-ordered perturbation theory. This is so because TOPT diagrams which differ only in the time-order of the physical interactions involved are all included in the one covariant perturbation theory graph. In particular, the two TOPT diagrams in Figure 1 arise from the same covariant perturbation theory graph. Hence, if equations for the $NN - \pi NN$ system are derived from this covariant perturbation theory diagrammatic expansion they will not suffer from the fundamental deficiency of the standard $NN - \pi NN$ equations.

Pursuing this approach to the $NN-\pi NN$ system also has the advantage that, provided care is taken in truncating the field theory, the equations will automatically be covariant, since they are derived from a covariant diagrammatic expansion. Furthermore, if integral equations are used to sum classes of diagrams containing infinitely many perturbation graphs, they generate amplitudes which are *non*-perturbative, even though the equations themselves may originally be derived from the perturbative diagrammatic expansion. Examples of this are the Schwinger-Dyson equations of a field theory which may be derived from a resummation of the original perturbation expansion of the theory, but in no way involve a truncation of this expansion at some order in the coupling constant of the theory. Such integral equations governing the $NN - \pi NN$ system will of necessity be four-dimensional, and hence the numerical solution of these equations in order to obtain experimental predictions is a challenging problem. However, until this problem is tackled it is not clear that the one-explicit-pion sector of the $NN - \pi NN$ system has been properly dealt with. Indeed, the standard theory of the $NN - \pi NN$ system, based on time-ordered perturbation theory, only approximates the fuller description based on a covariant diagrammatic expansion. Until it is clear that this approximation is an adequate one appeals to mechanisms beyond the one-pion sector to remedy the present disagreement between theory and experiment cannot be definitely upheld or overturned. The derivation of equations for the $NN - \pi NN$ system in the framework of covariant perturbation theory is therefore an important question at the heart of one of the oldest problems in theoretical nuclear physics.

The question then is: how are such equations to be derived? Given a Lagrangian one could use functional techniques to derive the relevant Schwinger-Dyson equations of the field theory, truncating the Schwinger-Dyson equation hierarchy by some approximation scheme, and so obtaining a set of four-dimensional coupled integral equations for the $NN - \pi NN - \pi\pi NN - \ldots$ system. In this paper we choose not to employ such a functional calculus technique, but instead use the Taylor method of classification-of-diagrams [20].

The Taylor method is a general one allowing the derivation of equations connecting the amplitudes obtained from diagrammatic expansions, and hence is ideally suited to the derivation of equations from the Feynman diagrammatic expansion of the amplitudes for the $NN - \pi NN$ system. Reference [21], henceforward known as paper I, presented a review of the Taylor method. That paper also pointed out that Taylor's method leads to double-counting when applied to certain covariant perturbation theory amplitudes. It was shown how the Taylor method could be modified in order to eliminate this double-counting, thus producing a technique by which double-counting-free covariant four-dimensional scattering equations may be derived for, not only the $NN - \pi NN$ system, but also other few-hadron systems, such as the $\pi N - \pi\pi N$ system and the problem of pion photoproduction.

This idea of applying Taylor's original method to the $NN - \pi NN$ system is not a new one. Three pairs of authors have already attempted to derive covariant four-dimensional equations for the $NN - \pi NN$ system using Taylor's method. Firstly, Avishai and Mizutani (AM), derived coupled covariant four-dimensional equations for the $NN - \pi NN$ system using the Taylor method [10]. However, Avishai and Mizutani failed to eliminate the double-counting which arises when the Taylor method is applied directly to the graphical expansion of a time-dependent perturbation theory. Consequently the equations they derived double-counted certain diagrams. Avishai and Mizutani then used Blankenbecler-Sugar [22] reduction in order to reduce their covariant four-dimensional integral equations to more manageable three-dimensional ones [23]. The reduced equations thus obtained are equivalent to the standard three-dimensional $NN - \pi NN$ equations AM (and others) had previously derived. The standard $NN - \pi NN$ equations do not contain any double-counting and thus the final set of equations used for numerical work by AM may be regarded as double-counting free. However, AM's "derivation" of these equations by a three-dimensional reduction of four-dimensional equations which themselves *do* contain double-counting is open to question.

Secondly, in 1985 Afnan and Blankleider (AB) derived a set of covariant $BB - \pi BB$ equations, in which the baryon B could be either a nucleon or a delta [17]. However, instead of using Taylor's original classification-of-diagrams scheme Afnan and Blankleider used a simplification of Taylor's method which they, together with Thomas and Rinat, had developed some years before [6,11]. (This simplification and its relation to Taylor's original work were discussed in paper I.) The equations thus obtained by Afnan and Blankleider were exactly those found by Avishai and Mizutani, except that the nucleon N in Avishai and Mizutani's theory was replaced throughout by a baryon B which could be either a nucleon or a delta [1]. The use of a diagrammatic technique similar to that used by Avishai and Mizutani, and

[1]There were terms present in AM's equations which were not included in AB's results, but the addition of these terms to AB's equations could have been effected with only minor changes to their argument.

the consequent derivation of similar equations naturally meant that Afnan and Blankleider's equations also contained double-counting if viewed in a four-dimensional framework. However, as in Avishai and Mizutani's case, this problem was never fully revealed, since the $BB - \pi BB$ equations were solved in a three-dimensional time-ordered perturbation theory, and so the double-counting was temporarily hidden, even though it was still present in the full four-dimensional theory.

Recently, Kvinikhidze and Blankleider (KB) have recognized the double-counting in these two derivations [24,25]. They have introduced a modification to Taylor's method as applied by AM and AB which allows them to derive a set of covariant four-dimensional equations for the $NN - \pi NN$ system which are free from double-counting. These equations are, apart from a minor point, equivalent to the ones derived here. However, this work differs from that of Ref. [24,25] in three main ways. Firstly, the method used here is rigorously based on the modification of Taylor's original classification scheme for an $m \to n$ amplitude in any perturbation theory as detailed in I. By contrast, KB have used a classification-of-diagrams scheme which is only loosely based on Taylor's original work. Secondly, our use of the full Taylor method allows us to exploit the true Lagrangian independence of that technique. Hence here we do not specify the Lagrangian to be used in the description of the $NN - \pi NN$ system, while KB restricted themselves to the case of a ϕ^3 field-theory. Finally, we go beyond KB's work in casting our equations in a form convenient for computation by deriving a set of coupled equations for two-fragment amplitudes in the $NN - \pi NN$ system.

This work [2] proceeds as follows. In Section II Green's functions and amplitudes of the field theory are defined while Section III provides a brief summary of the Taylor method of classification of diagrams. In Section IV the NN amplitude T_{NN} is discussed and is found to depend on the fully-connected two-particle irreducible $NN \to \pi NN$ amplitude, $F^{(2)\dagger}$. Section V is therefore devoted to deriving an equation for $F^{(2)\dagger}$, a process which involves eliminating the double-counting that occurs in the equation found when the Taylor method is applied directly to $F^{(2)\dagger}$. Section VI then outlines the removal of double-counting from the equation for the two-particle irreducible amplitude $T^{(2)}_{NN}$. Once in possession of double-counting-free equations for the amplitudes involved we proceed in Section VII to derive a set of coupled equations for the $NN - \pi NN$ system. Section VIII then explains how these equations may be anti-symmetrized in order to obtain the physical amplitudes. Finally, Section IX gives suggestions for the specification of the input amplitudes and explains how physics beyond the one-pion sector can be implicitly included in the framework developed in Sections II–VIII.

The work concludes with four appendices containing details of arguments to do with the dressing of the propagators, the amplitudes which are the input to the equations, and details of the double-counting removal performed in Section VI.

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[2] A summary of these results was presented at the 14$^{\text{th}}$ International Conference on Few Body Problems in Physics [26].

II. THE FIELD THEORY OF NUCLEONS AND PIONS

In this section we define the field theory of nucleons and pions used in this work by discussing the Lagrangian to be used and defining the Green's functions and amplitudes of the theory.

A. The Lagrangian

The Taylor method is essentially a topological classification of the diagrams in a perturbation expansion. Therefore, it depends only on the topology of the diagrams which can be generated, and not on the details of the particular Lagrangian under consideration. Consequently, in this work we do not specify the precise form of the Lagrangian to be used. We merely assume that the Lagrangian density describes a system of nucleons and pions and so takes the form:

$$\mathcal{L} = \mathcal{L}_{\mathcal{D}} + \mathcal{L}_{\phi} + \mathcal{L}_{int}, \tag{1}$$

where:

$$\mathcal{L}_D(x) = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x), \tag{2}$$

$$\mathcal{L}_\phi(x) = \frac{1}{2}(\partial_\mu\vec{\phi}(x)\cdot\partial^\mu\vec{\phi}(x) - m_\pi^2\vec{\phi}(x)\cdot\vec{\phi}(x)), \tag{3}$$

where ψ is the nucleon field and $\vec{\phi}$ the pion field, which is an isovector.

At this stage we do not actually need to assume *anything* about the structure of $\mathcal{L}_{int}$ in order to apply the Taylor method. However, for concreteness the reader may wish to assume that $\mathcal{L}_{int}$ is a sum of a bare $NN\pi$ vertex, proportional to $\bar{\psi}\phi\psi$ and a contact term, proportional to $\bar{\psi}\phi\phi\psi$. As will be seen below, our derivation never demands that we say more than this. Bare form factors for these two kinds of vertices could be included in the Lagrangian if we wished. Such form factors would be a natural place to include information from QCD about the structure of the nucleons and pions in our field theory. However, for the present we leave these details to one side.

Note also that the Lagrangian independence of the Taylor method makes it relatively easy to return at a later stage and add other mesons and baryons to the field theory. The addition of extra particles to the theory merely complicates the resulting equations without changing the spirit of the derivation. However, as a first step in this paper we have derived the results for a field theory in which the only quanta are nucleons and pions.

B. Green's functions and amplitudes

The coordinate space Green's function for the transition from a j-nucleon, $m-j$-pion state to a j'-nucleon, $n-j'$-pion state is defined by:

$$\begin{aligned} &G^{(j',j)}_{n\leftarrow m}(x'_1,\ldots,x'_{j'},x'_{j'+1},\ldots,x'_n;x_1,\ldots,x_j,x_{j+1},\ldots,x_m) = \\ &\quad\langle 0|T(\psi(x'_1)\ldots\psi(x'_{j'})\phi(x'_{j'+1})\ldots\phi(x'_n)\overline{\psi}(x_1)\ldots\overline{\psi}(x_j)\phi(x_{j+1})\ldots\phi(x_m))|0\rangle, \end{aligned} \tag{4}$$

where $|0\rangle$ is the vacuum state, T is the time-ordering operator, $\psi(\overline{\psi})$ is the nucleon annihilation (creation) operator, and ϕ is the pion operator, whose isospin index has been suppressed.

This Green's function may then be Fourier transformed in order to obtain the momentum-space Green's function:

$$G^{(j',j)}_{n\leftarrow m}(p'_1,\ldots,p'_{j'},p'_{j'+1},\ldots,p'_n;p_1,\ldots,p_j,p_{j+1},\ldots,p_m). \tag{5}$$

From this momentum-space Green's function the amplitude:

$$A^{(j',j)}_{n\leftarrow m}(p'_1,\ldots,p'_{j'},p'_{j'+1},\ldots,p'_n;p_1,\ldots,p_j,p_{j+1},\ldots,p_m) \tag{6}$$

is obtained via the LSZ reduction [27] formula:

$$\begin{aligned}A^{(j',j)}_{n\leftarrow m}(p'_1,\ldots,p'_n;p_1,\ldots,p_m) = Z_N^{-\frac{j'}{2}} Z_\pi^{-\frac{n-j'}{2}} d_N^{-1}(p'_1)\ldots d_N^{-1}(p'_{j'}) d_\pi^{-1}(p'_{j'+1})\ldots d_\pi^{-1}(p'_n)\\ G^{(j',j)}_{n\leftarrow m}(p'_1,\ldots,p'_n;p_1,\ldots,p_m) d_N^{-1}(p_1)\ldots d_N^{-1}(p_j) d_\pi^{-1}(p_{j+1})\ldots d_\pi^{-1}(p_m) Z_N^{-\frac{j}{2}} Z_\pi^{-\frac{m-j}{2}}\end{aligned} \tag{7}$$

where d_N and d_π are the free single-nucleon and single-pion propagator in momentum space, given by the formulae:

$$d_N(p) = \frac{i}{\not p - m_N}, \tag{8}$$

$$d_\pi(k) = \frac{i}{k^2 - m_\pi^2}. \tag{9}$$

Note that we are using fully dressed propagators in order to do the reduction here and so Z_N and Z_π are the wave function renormalizations for the nucleon and pion respectively, while m_N and m_π are the dressed nucleon and pion masses. The Green's function $G^{(j',j)}_{n\leftarrow m}$, and therefore the amplitude $A^{(j',j)}_{n\leftarrow m}$, may be expressed as the sum of a perturbation series of Feynman diagrams.

Note that although all the formulae in this section are written for identical particles, in this paper we (at first) consider only the amplitudes for distinguishable particles. The physically correct indistinguishable-particle amplitudes may then be obtained from these distinguishable-particle amplitudes by the usual symmetrization and anti-symmetrization processes. This procedure is fully justified and implemented in Section VIII.

III. THE TAYLOR METHOD OF CLASSIFICATION-OF-DIAGRAMS

A. Taylor's original method

Taylor's method of classification-of-diagrams then provides a scheme for classifying all of the diagrams which contribute to any amplitude $A^{(j',j)}_{n\leftarrow m}$ according to their topology. In this paper we concentrate on the topological structure of the diagrams in the s-channel [3], although it is possible to apply the Taylor procedure in other channels instead of, or

[3]We use the notation $s-$,$t-$ and u-channel throughout this paper, intending it in the sense defined by Mandelstam [28,29].

as well as, applying it in this channel. Note that because of this focus on the s-channel structure, unless otherwise specified, the irreducibilities given for amplitudes and diagrams are s-channel irreducibilities. Note also that from now on we suppress the indication of the number of nucleons and pions present in the initial and final states, in order to simplify the notation.

Having noted these points the Taylor scheme then requires the following definitions:

Definition 1 (r-cut) *An r-cut is an arc which separates initial from final states and intersects exactly r-lines, at least one of which must be an internal line. If all of the r lines cut are internal lines then the cut is called an* internal r-cut*.*

Note that in writing this definition we assume that all perturbation diagrams are represented in a two-dimensional plane. We do allow the lines in any diagram to "jump over" one another: two lines do not have to meet at an interaction vertex whenever they intersect. By contrast, a cut is defined to intersect all the lines it meets: it may not jump over any of them. (Other definitions of an r-cut, which do not assume that the diagrams are represented in the plane, may be composed but it is the above definition which Taylor himself used.)

Definition 2 (r-particle irreducibility) *A diagram is called r-particle irreducible if, for all integers $0 \leq k \leq r$, no k-cut may be made on it. An amplitude is called r-particle irreducible if all diagrams contributing to it are r-particle irreducible.*

The r-cuts which may be made on a particular diagram are now divided as follows: if an r-cut is not internal it is called "initial" if it intersects at least one initial-state but no final-state line; "final" if it intersects at least one final-state, but no initial-state line, and "mixed" if it intersects both initial and final-state lines.

Once these definitions are made any diagram contributing to the connected s-channel $(r-1)$-particle irreducible $(r-1\text{PI})$ $m \to n$ amplitude, $A^{(r-1)}_{n\leftarrow m}$, may be placed in one of the five classes C_1 to C_5. The class a particular graph belongs to is determined by the r-cuts which may be made on it, as follows:

- C_1: No r-cut may be made on the diagram, i.e. it is r-particle irreducible;
- C_2: At least one internal r-cut may be made on the diagram and no mixed or final r-cut may be made.
- C_3: At least one initial r-cut is possible on the diagram, but all other types of cut are not possible;
- C_4: At least one mixed r-cut is possible, but a final r-cut is impossible;
- C_5: At least one final r-cut may be made.

This classification is motivated by where the "latest" r-cut may be made on a particular graph. If it is an internal cut then the graph must be placed in C_2, but if it is an external cut then the graph is placed in one of C_3–C_5 depending on what sort of cut that "latest" r-cut may be.

Because each graph contributing to A belongs to one and only one of these classes it follows that A may be expressed as the sum of the five expressions found by summing the individual classes separately.

The sum of class C_1 is clearly the connected s-channel r-particle irreducible $m \to n$ amplitude, $A^{(r)}_{n \leftarrow m}$. Each of the classes C_2 to C_5 may be summed using the following result, which is known as the last internal cut lemma (LICL):

Lemma 1 (Last Internal Cut) *Any $(r-1)$-particle irreducible diagram which admits an internal r-cut has a unique internal r-cut which is nearest to the final state.*

For a proof of this result see paper I. Explicit results for the sum of classes C_3–C_5 are also given there. Note also the following results, more fully explained in paper I:

1. The Last Internal Cut Lemma (LICL) may not be applied directly when constructing the sum of classes C_3–C_5;

2. In order to correctly sum classes C_3, C_4 and C_5 one not only needs to restrict the s-channel cut-structure of the amplitudes in the sums of these classes, but also to place constraints upon the cut structure of these amplitudes in channels other than the s-channel. (This is a point whose full implications were apparently not realized by Taylor himself.) However, many of these additional restrictions prove to be trivially satisfied by the amplitudes under consideration, due to the stringent nature of the s-channel irreducibilities enforced by the LICL.

B. The double-counting problem and its solution

As was also explained in paper I, the adoption of Taylor's procedure for summing the classes C_3 to C_5 leads to the overcounting of certain diagrams, as follows.

In the case where $n \leq r$ there are complications with the use of the last internal cut lemma in classes C_4 and C_5. If $m < r$ similar difficulties arise in classes C_3 and C_4. Taylor's solution to these difficulties is to split any class C in which such a problem occurs into sub-classes S. These sub-classes are defined in such a way that the last internal cut lemma may be applied to the diagrams within them without complications, and consequently their sum may be constructed correctly. But, Taylor attempts to then construct the sum of C by summing over all possible sub-classes S. This is correct, provided that the sub-classes are defined so as to be completely disjoint. However, with Taylor's definition of sub-classes of C the sub-classes are *not* disjoint. Certain diagrams belong to more than one sub-class and so are double-counted. In paper I the details of this problem were discussed and specific examples given. It was then shown how to avoid the double-counting problem by using the following procedure to correctly construct the sum of any class C in which double-counting is a possibility.

The procedure involves considering each sub-class S of C in turn. The sub-classes are added to a "running tally" of C, one by one. However, a particular sub-class S may only be added to the running tally after all of the diagrams within S which have already been included in the "running tally" have been eliminated from S.

We express this mathematically by saying that if C is the union of n sub-classes, i.e. $C = \cup_{i=1}^{n} S_i$, then the correct sum of C is:

$$\sum_{i=1}^{n} S_i - \sum_{\substack{i,j=1 \\ i<j}}^{n} S_i \cap S_j + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} S_i \cap S_j \cap S_k - \ldots . \tag{10}$$

The difficult part of the application of this formula is the discovery of which diagrams are in the intersections $S_i \cap S_j$, $S_i \cap S_j \cap S_k$, etc.. Because of the way the sub-classes are defined any diagram which is in a set $S_i \cap S_j$ must admit two possible "unique" latest r-cuts, each cutting a different set of external lines. Consequently, the diagrammatic content of the set $S_i \cap S_j$ may be found by examining the full s-channel cut-structure of the amplitude that is the sum of S_i and so determining which portions of that amplitude have a cut-structure which means that they also belong to sub-class S_j. In covariant perturbation theory, or, for that matter, in any time-dependent perturbation theory, the result of this examination will depend, not only on the s-channel cut-structure of the sub-amplitudes making up the amplitude representing S_i, but also on their cut-structure in other channels. Therefore, once the double-counting is eliminated by doing the subtraction recommended in Eq. (10) certain constraints are effectively placed on the cut-structure of these sub-amplitudes in channels other than the s-channel. Thus, in spite of the fact that we consider only the s-channel cut-structure of the full amplitude, the elimination of double-counting effectively places constraints on the cut-structure of the sub-amplitudes in channels other than the s-channel.

This modification to the Taylor method provides a way of deriving covariant four-dimensional scattering equations which are free of double-counting for any system in which it is possible to obtain a graphical perturbation expansion for the relevant amplitudes. The only *caveat* is that the particles involved must be fully dressed before the Taylor method is applied. In Appendix A we explain how this is accomplished for a system of nucleons and pions. Also, in Appendix B the Taylor method is used to derive equations for the amplitudes which are input to the $NN - \pi NN$ equations: the πN amplitude and the πNN vertex. In the next section we demonstrate how the Taylor method as outlined here would be applied to the NN t-matrix, T_{NN}.

IV. THE $N-N$ AMPLITUDE, PART 1

Now consider the $N-N$ amplitude, T_{NN}. In this work we are interested in particle-particle scattering, and so we may consider the particles in the initial and final states to both be nucleons, not anti-nucleons. Consequently, the baryon number in any intermediate state must be two, and so the amplitude T_{NN} can only have an s-channel one-particle irreducible part $T_{NN}^{(1)}$. Furthermore, since, by assumption, both nucleons are fully-dressed, this amplitude must be connected. Taylor's method may therefore be applied to this $T_{NN}^{(1)}$ in order to derive an integral equation for NN scattering.

So consider any diagram contributing to $T_{NN}^{(1)}$. Any cuts which could be made to place this diagram in C_3–C_5 would merely expose dressing contributions on the external nucleon legs. Since all particles are fully dressed such dressing contributions must be zero, and hence classes C_3–C_5 are empty. Meanwhile, class C_1 sums to the connected two-particle irreducible NN amplitude $T_{NN}^{(2)}$, and class C_2 yields, using the LICL, the product $T_{NN}^{(2)} d_1 d_2 T_{NN}^{(1)}$ where d_1

and d_2 are the fully-dressed free propagators for nucleons 1 and 2. Hence, Taylor's method gives:

$$T_{NN}^{(1)} = T_{NN}^{(2)} + T_{NN}^{(2)} d_1 d_2 T_{NN}^{(1)}. \tag{11}$$

Eq. (11) is, of course, a Bethe-Salpeter equation for $T_{NN}^{(1)}$, with $T_{NN}^{(2)}$ the kernel of the equation, containing the diagrams to be iterated in order to produce the full amplitude. However, above the pion-production threshold $T_{NN}^{(2)}$ contains part of the effects of inelasticity. (The fully-dressed nucleon propagators also contain some inelasticity effects.) Consequently, if the NN sector is to be modelled correctly it is necessary that we examine the $NN\pi$ cut structure of $T_{NN}^{(2)}$. However, if this three-particle cut structure of $T_{NN}^{(2)}$ is to be examined via the Taylor method we have $m = n = 2$ and $r = 3$, so the conditions for double-counting in classes C_3, C_4 and C_5 are satisfied. Therefore, we must proceed with caution.

Applying Taylor's method directly to $T_{NN}^{(2)}$ means that we sum each class individually, as follows:

C_1: The sum of class C_1 is clearly the connected 3PI $2 \to 2$ amplitude, $T_{NN}^{(3)}$.

C_2: Using the last-internal cut lemma we may show that the sum of class C_2 is:

$$F^{(3)} d_1 d_2 d_\pi F^{(2)\dagger}, \tag{12}$$

where $F^{(r)}$ is the connected r-particle irreducible $NN\pi \to NN$ amplitude, and $F^{(r)\dagger}$ is the connected r-particle irreducible $NN \to NN\pi$ amplitude. Now, C_2 may not contain any diagrams admitting a three-cut which involves a final-state line. It might be thought that in order to enforce this condition in Eq. (12) certain restrictions would need to be placed on $F^{(3)}$. However, when these extra conditions are examined closely it is found that they are all satisfied trivially, due to the s-channel three-particle irreducibility of $F^{(3)}$. (This is, in fact, a special case of a more general result, that in C_2 such extra conditions are always unnecessary, because the cuts which may intersect a final-state line and so place diagrams in C_4 or C_5 are precluded by the s-channel cut-structure imposed on the amplitudes involved.)

C_3: As explained in paper I, the sum of class C_3 must be constructed with care. This is because the LICL applies only to internal cuts, and so cannot be used directly on diagrams in C_3, since, by definition, diagrams in C_3 may have cuts which intersect both internal and external lines.

In order to overcome this problem Taylor constructs two sub-classes of C_3, one of which contains those diagrams in which the "last" cut cuts the line $N1$, and the other of which contains those diagrams in which the "last" cut cuts the line $N2$. (Here the sense in which the cut we are talking about is the "last" cut was defined in paper I.) Taylor calls the first sub-class $C_3^{\{N1\}}$ and the second $C_3^{\{N2\}}$. Using the LICL he constructs their sums:

$$C_3^{\{N1\}} = F^{(3)} d_2 d_\pi f^{(1)\dagger}(1), \tag{13}$$

$$C_3^{\{N2\}} = F^{(3)} d_1 d_\pi f^{(1)\dagger}(2), \tag{14}$$

where ${f^{(1)}}^\dagger(i)$ is the 1PI πNN vertex corresponding to the emission of a pion by nucleon $\bar{i}$ while nucleon i is a spectator. I.e., $\bar{i}$ is defined by:

$$\bar{i} = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2. \end{cases} \tag{15}$$

(Note that once again it might be expected that conditions must be placed on $F^{(3)}$ in order to ensure that no diagram which actually belongs in C_4 or C_5 is included in C_3. And, indeed, as we shall see below, in some cases the amplitudes used in C_3 do have to be restricted. But, in the case under discussion here the s-channel 3PI of $F^{(3)}$ guarantees that only diagrams which belong in C_3 are included in the sums (13) and (14).)

Taylor then claims that the sum of C_3 is merely the sum of the two sub-class sums (13) and (14). But this is false, since certain diagrams will have two distinct possible "latest" cuts, one cutting $N1$ and one cutting $N2$. (See, e.g. Figure 2.) These diagrams will belong to both sub-classes, and so will be double-counted in such a summation. In order to correct this double-counting we apply Eq. (10) so as to obtain the corrected sum of class C_3:

$$C_3 = \sum_{i=1}^{2} F^{(3)} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i) - D_3, \tag{16}$$

where:

$$D_3 = C_3^{\{N1\}} \cap C_3^{\{N2\}}, \tag{17}$$

has been subtracted in order to remove the double-counting.

C_4: Adopting a similar approach to that used to sum C_3, we follow Taylor, and divide C_4 into two sub-classes: $C_4^{\{N2'\}\{N1\}}$, which contains all those diagrams in which the "latest" cut cuts lines $N2'$ and $N1$, and $C_4^{\{N1'\}\{N2\}}$, which contains all those diagrams in which the "latest" cut cuts lines $N1'$ and $N2$. The last internal cut lemma may then be employed in order to write:

$$C_4^{\{N2'\}\{N1\}} = f^{(2)}(2) d_\pi \tilde{f}^{(1)\dagger}(1), \tag{18}$$

$$C_4^{\{N1'\}\{N2\}} = f^{(2)}(1) d_\pi \tilde{f}^{(1)\dagger}(2). \tag{19}$$

Note that in order to stop diagrams which should be in C_5 straying into C_4 we have had to restrict the amplitude ${f^{(1)}}^\dagger$ used in (18) and (19) to only contain diagrams which are 2PI in the $N' \leftarrow N + \pi'$-channel. We have denoted the resulting amplitude by $\tilde{f}^{(1)\dagger}$. If the restriction on ${f^{(1)}}^\dagger$ is not enforced then the three-cuts depicted in Figure 3 may be made on certain diagrams in C_4, indicating that these diagrams actually belong in C_5, not C_4

If we now apply Eq. (10) we find that the correct sum of C_4 is:

$$C_4 = \sum_{i,j=1}^{2} f^{(2)}(j) d_\pi \tilde{f}^{(1)\dagger}(i) \bar{\delta}_{ij} - D_4, \tag{20}$$

with:

$$D_4 = C_4^{\{N2'\}\{N1\}} \cap C_4^{\{N1'\}\{N2\}}. \tag{21}$$

C_5: Once more this class is split into sub-classes. This time the sub-classes are $C_5^{\{N1'\}}$ and $C_5^{\{N2'\}}$, which contain, respectively, all diagrams in which the "latest" cut cuts lines $N1'$ and $N2'$. The LICL gives:

$$C_5^{\{Nj'\}} = f^{(2)}(j) d_{\bar{j}} d_\pi F^{(2)^\dagger}, \tag{22}$$

for $j = 1, 2$. Consequently we have:

$$C_5 = \sum_{j=1}^{2} f^{(2)}(j) d_\pi d_{\bar{j}} F^{(2)^\dagger} - D_5, \tag{23}$$

with:

$$D_5 = C_5^{\{N1'\}} \cap C_5^{\{N2'\}}. \tag{24}$$

If we combine the above results for the sums of the individual classes we find:

$$\begin{aligned} T_{NN}^{(2)} = T_{NN}^{(3)} &+ F^{(3)} d_1 d_2 d_\pi (F^{(2)^\dagger} + \sum_i f^{(1)^\dagger}(i) d_i^{-1}) - D_3 \\ &+ \sum_j f^{(2)}(j) d_{\bar{j}} d_\pi (\sum_i \bar{\delta}_{ij} \tilde{f}^{(1)\dagger}(i) d_i^{-1} + F^{(2)^\dagger}) - D_4 - D_5 \end{aligned} \tag{25}$$

where here, and throughout the rest of the paper, the sums over i and j are understood to run over $i, j = 1, 2$. This equation, but without the restriction on $f^{(1)^\dagger}$ in class C_4 and the subtractions for double-counting, was also derived by Avishai and Mizutani [10] and Afnan and Blankleider [11,17]. Below we shall calculate the precise values of the subtraction terms D_3, D_4 and D_5. But, in order to achieve this and so derive an equation for $T_{NN}^{(2)}$ we will need to know the structure of the amplitudes in Eq. (25).

Firstly, consider the amplitudes $T_{NN}^{(3)}$ and $F^{(3)}$. Explicit examination of the structure of these amplitudes would involve exposing $NN\pi\pi$ intermediate states. Since our primary concern here is the derivation of equations which treat the $NN\pi$ intermediate states of the theory correctly, we argue that these amplitudes may be safely neglected or approximated. The equations resulting from this approximation will then satisfy NN and $NN\pi$ unitarity, but not $NN\pi\pi$ unitarity. As a first approximation we set $F^{(3)}$ and $T_{NN}^{(3)}$ to zero, in order to make our equations as simple as possible. This approximation is, in fact, exact in the case of an $N\pi$ interaction Lagrangian containing only a πNN vertex.

The use of this approximation means that the sum of both sub-classes of C_3 is zero, and so $D_3 = 0$. Therefore, given this assumption, $T_{NN}^{(2)}$ obeys:

$$T_{NN}^{(2)} = \sum_j f^{(2)}(j) d_{\bar{j}} d_\pi F^{(2)\dagger} + \sum_{ij} \bar{\delta}_{ij} f^{(2)}(j) d_\pi \tilde{f}^{(1)\dagger}(i) - D_4 - D_5, \tag{26}$$

with D_4 and D_5 given by equations (21) and (24) respectively. The amplitude $f^{(2)}$ which appears in (26) can be extracted from the model Lagrangian, as demonstrated in Appendix B. As far as the amplitude $\tilde{f}^{(1)\dagger}$ is concerned, for the moment we merely observe that it must contain some subset of the diagrams which are summed by solving Eq. (B2) for $f^{(1)}$. Therefore the only amplitude whose structure remains to be investigated is $F^{(2)\dagger}$.

V. THE TWO-PARTICLE IRREDUCIBLE, $NN \to NN\pi$ AMPLITUDE, $F^{(2)\dagger}$

Our examination of the two-particle irreducible NN amplitude $T_{NN}^{(2)}$ revealed that the equation it obeyed involved the connected 2PI $NN \to NN\pi$ amplitude $F^{(2)\dagger}$. In this section we examine this amplitude using the original Taylor method, and eliminate the double-counting inherent in that method, using the modification to the Taylor method described in Section III B above.

In this case the sum of the Taylor classes and sub-classes is as follows:

$$C_1 = F^{(3)\dagger} \tag{27}$$

$$C_2 = M^{(3)} d_1 d_2 d_\pi F^{(2)\dagger} \tag{28}$$

$$C_3^{\{N1\}} = M_1^{(3)} d_2 d_\pi f^{(1)\dagger}(1) \tag{29}$$

$$C_3^{\{N2\}} = M_2^{(3)} d_1 d_\pi f^{(1)\dagger}(2) \tag{30}$$

$$C_4^{\{N1'\}\{N2\}} = t_{\pi N}^{(2)}(1) d_\pi f^{(1)\dagger}(2) \tag{31}$$

$$C_4^{\{N2'\}\{N1\}} = t_{\pi N}^{(2)}(2) d_\pi f^{(1)\dagger}(1) \tag{32}$$

$$C_4^{\{\pi'\}\{N1\}} = T_{NN}^{(2)} d_\pi f^{(1)\dagger}(1) \tag{33}$$

$$C_4^{\{\pi'\}\{N2\}} = T_{NN}^{(2)} d_\pi f^{(1)\dagger}(2) \tag{34}$$

$$C_5^{\{N1'\}} = t_{\pi N}^{(2)}(1) d_2 d_\pi F^{(2)\dagger} \tag{35}$$

$$C_5^{\{N2'\}} = t_{\pi N}^{(2)}(2) d_1 d_\pi F^{(2)\dagger} \tag{36}$$

$$C_5^{\{\pi'\}} = T_{NN}^{(2)} d_1 d_2 F^{(2)\dagger}, \tag{37}$$

where we have labeled the incoming nucleons $N1$ and $N2$, the outgoing nucleons $N1'$ and $N2'$, the outgoing pion π', and $t_{\pi N}^{(2)}(i)$ is the 2PI $\pi - N$ t-matrix with nucleon i as a spectator. Here, $M^{(3)}$, $M_1^{(3)}$ and $M_2^{(3)}$ are all connected 3PI $NN\pi \to NN\pi$ amplitudes. However, constraints beyond simple s-channel 3PI have been placed on the amplitudes $M_1^{(3)}$ and $M_2^{(3)}$. These constraints mean that $M_1^{(3)}$ and $M_2^{(3)}$ are different and neither is equal to the sum of all s-channel 3PI $NN\pi \to NN\pi$ diagrams, $M^{(3)}$. It is found that in order to stop diagrams actually belonging to C_4 being mistakenly included in C_3 we must define $M_i^{(3)}$ to be 1PI in both the channels:

$$Ni' + Ni \leftarrow N\bar{i}' + N\bar{i} + \pi' + \pi$$

and:

$$\pi' + Ni \leftarrow N\bar{i}' + N\bar{i} + Ni' + \pi.$$

It might appear that conditions beyond s-channel 3PI would also have to be placed on the $M^{(3)}$ appearing in C_2, in order to prevent diagrams from C_4 and C_5 being included there too. However, in the case under consideration in this section the possibility of any cut placing a diagram which is summed in C_2 above in C_4 or C_5 is precluded by the s-channel 3PI of $M^{(3)}$. Therefore, as was the case with the amplitude $F^{(3)}$ used in C_2 and C_3 in the previous section, no constraints beyond s-channel 3PI are necessary.

Note that once these restrictions are imposed on $M_i^{(3)}$ the problem discussed by Kvinikhidze and Blankleider [25], where diagrams hidden in $M^{(3)} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i)$ actually represent processes included in C_4, does not occur, since the diagrams in C_4 which give rise to these difficulties are specifically excluded from $C_3^{\{N1\}}$ and $C_3^{\{N2\}}$ by the above irreducibility conditions. Therefore we may safely conclude that all diagrams in C_2 and C_3 will *only* represent processes involving some piece of the connected s-channel 3PI $NN\pi \to NN\pi$ amplitude. In this paper any such process is said to involve a "pure three-body force". In the calculation here we assume that to a first approximation these pure three-body forces are negligible, by an appeal to the same argument which resulted in us neglecting the amplitudes $F^{(3)}$ and $T^{(3)}$ above. Consequently, we assume that the sum of classes C_2 and C_3 is zero.

The inclusion of a three-body force, $M^{(3)}$ in our calculation, would result in considerable changes to the ensuing argument, since at numerous stages during our discussion below we have appealed to the absence of a three-body force as a reason for eliminating certain double-counting subtraction terms. Nevertheless, integral equations for the amplitudes involved may still be derived, even if $M^{(3)} \neq 0$. In that case, however, it is considerably harder to see how such equations may be manipulated into a form for practical calculation.

Once these observations have been made it is a simple matter to sum the above expressions for the sub-classes in order to obtain the following equation for ${F^{(2)}}^\dagger$:

$$ {F^{(2)}}^\dagger = \sum_\alpha t^{(2)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi \Big(\sum_i \bar{\delta}_{i\alpha} {f^{(1)}}^\dagger(i) d_i^{-1} + {F^{(2)}}^\dagger\Big) - D_4 \qquad (38)$$

with the sum over α running over $\alpha = 1, 2, 3$ here and throughout the rest of this work; $t^{(2)}(\alpha)$ being defined by:

$$ t^{(2)}(\alpha) = \begin{cases} t_{\pi N}^{(2)}(i) & \text{if } \alpha = i = 1, 2 \\ T_{NN}^{(2)} & \text{if } \alpha = 3. \end{cases} \qquad (39)$$

and:

$$ \begin{aligned} D_4 = {} & C_4^{\{\pi'\}\{N1\}} \cap C_4^{\{\pi'\}\{N2\}} + C_4^{\{N2'\}\{N1\}} \cap C_4^{\{\pi'\}\{N2\}} + C_4^{\{N1'\}\{N2\}} \cap C_4^{\{\pi'\}\{N1\}} \\ & + C_4^{\{N2'\}\{N1\}} \cap C_4^{\{\pi'\}\{N1\}} + C_4^{\{N1'\}\{N2\}} \cap C_4^{\{\pi'\}\{N2\}} + C_4^{\{N1'\}\{N2\}} \cap C_4^{\{N2'\}\{N1\}}, \end{aligned} \qquad (40)$$

included in order to eliminate the double-counting in C_4, as is prescribed in Eq. (10). Note three things about this result:

1. The D_4 here is obviously different to the D_4 defined by Eq. (21) in the previous section, just as the sums of the Taylor classes in this section are different since we are dealing with a different amplitude here.

2. The maximum number of final-state lines cut by any three-cut is $s_f = 1$. Hence the condition $n \geq 2s_f$ holds and this condition is sufficient to guarantee the absence of double-counting in C_5. (See paper I for details.)

3. All three-sub-class intersections are zero in this particular case, thus D_4 is given by the sum of the two-sub-class intersections listed here.

We now calculate the six two-set intersections involved in D_4. In order to do this we note that any diagram which is in both $C_4^{\{A'\}\{B\}}$ and $C_4^{\{E'\}\{F\}}$ must admit two "latest" three-cuts, the first of which cuts the three lines A', B and one other, and the second of which cuts the lines E', F and one other. Consequently, the intersection of $C_4^{\{A'\}\{B\}}$ and $C_4^{\{E'\}\{F\}}$ may be found by investigating which diagrams contributing to the sum $C_4^{\{A'\}\{B\}}$ admit a three-cut involving E', F and one other particle.

Looking at the sum of $C_4^{\{\pi'\}\{N1\}}$ we see that no three-cut involving π', $N2$ and one other particle is possible. Therefore we conclude:

$$C_4^{\{\pi'\}\{N1\}} \cap C_4^{\{\pi'\}\{N2\}} = 0. \tag{41}$$

Similarly:

$$C_4^{\{N2'\}\{N1\}} \cap C_4^{\{\pi'\}\{N1\}} = 0, \tag{42}$$

and:

$$C_4^{\{N1'\}\{N2\}} \cap C_4^{\{\pi'\}\{N2\}} = 0. \tag{43}$$

Now consider $C_4^{\{N2'\}\{N1\}}$. Upon examining the diagrammatic sum of $C_4^{\{N2'\}\{N1\}}$ we observe that there are a number of diagrams contributing to the sum in which a three-cut involving π' and $N2$ may be made (see Fig. 4). Such a cut may be made in the portion of $C_4^{\{N2'\}\{N1\}}$ which sums to:

$$v_{\pi N}^X(2) d_\pi f^{(1)^\dagger}(1), \tag{44}$$

where $v_{\pi N}^X$ is the u-channel 1PR part of $t_{\pi N}^{(2)}$. In Appendix B it is shown that the part of $t_{\pi N}^{(1)}$ which is one-particle reducible in the s-channel is the pole diagram, $f^{(1)^\dagger} d_N f^{(1)}$. By changing the argument of Appendix B to apply to the u-channel instead of the s-channel it may be shown that $v_{\pi N}^X$ is merely the crossed term:

$$v_{\pi N}^X = f^{(1)} d_N f^{(1)^\dagger}. \tag{45}$$

Therefore,

$$C_4^{\{N2'\}\{N1\}} \cap C_4^{\{\pi'\}\{N2\}} = v^X(2) d_\pi f^{(1)^\dagger}(1) \tag{46}$$

$$= v_{OPE} d_1 f^{(1)^\dagger}(2), \tag{47}$$

Note that here v_{OPE} is the full one-pion exchange potential:

$$v_{OPE} = f^{(1)}(2) d_\pi f^{(1)^\dagger}(1). \tag{48}$$

This argument makes it explicit how the term (46) comes to be in two sub-classes. It is this inclusion of (46) in two different sub-classes which leads to the double-counting pointed out by Kowalski et al. [30].

Naturally exactly the same argument, but with the roles of $N1$ and $N2$ reversed, may be used to show that:

$$C_4^{\{N1'\}\{N2\}} \cap C_4^{\{\pi'\}\{N1\}} = v^X(1) d_\pi f^{(1)^\dagger}(2) \tag{49}$$

$$= v_{OPE} d_2 f^{(1)^\dagger}(1). \tag{50}$$

The identification in the last line may be made because the one-pion exchange potential may be written as:

$$v_{OPE} = f^{(1)}(1) d_\pi f^{(1)^\dagger}(2), \tag{51}$$

since we are working in a time-dependent perturbation theory, and so:

$$f^{(1)^\dagger} = f^{(1)}. \tag{52}$$

Note that in order for Eq. (52) to hold all particles involved must be fully dressed, so that $f^{(1)}$ is one-particle irreducible in all channels.

Finally we attempt to construct the intersection of $C_4^{\{N1'\}\{N2\}}$ and $C_4^{\{N2'\}\{N1\}}$. Any diagram which is in $C_4^{\{N1'\}\{N2\}}$ and admits an s-channel three-cut involving $N2'$, $N1$ and one other particle will also be in $C_4^{\{N2'\}\{N1\}}$. As is shown in Fig. 5 such a cut is possible if the amplitude $t_{\pi N}^{(2)}$ used in $C_4^{\{N1'\}\{N2\}}$ is one-particle reducible in the t-channel. Therefore in models including diagrams such as Fig. 6 the intersection will be given by:

$$C_4^{\{N1'\}\{N2\}} \cap C_4^{\{N2'\}\{N1\}} = v_{\pi N}^\rho(1) d_\pi f^{(1)^\dagger}(2), \tag{53}$$

where $v_{\pi N}^\rho$ is the t-channel one-particle reducible part of the s-channel 2PI π-N amplitude. However, in our model pions do not interact with any particle other than nucleons, and so the $\pi - N$ t-matrix is automatically one-particle irreducible in the t-channel. Consequently, the cut shown in Fig. 5 is simply not possible, leading to:

$$C_4^{\{N1'\}\{N2\}} \cap C_4^{\{N2'\}\{N1\}} = 0. \tag{54}$$

Therefore we have:

$$D_4 = \sum_{ij} f^{(1)}(i) d_j f^{(1)^\dagger}(i) d_\pi f^{(1)^\dagger}(j) \bar{\delta}_{ij} \tag{55}$$

$$= \sum_{ij} v^X(i) d_\pi f^{(1)^\dagger}(j) \bar{\delta}_{ij} \tag{56}$$

$$= \sum_{ij} v_{OPE} d_j f^{(1)^\dagger}(i) \bar{\delta}_{ij} \tag{57}$$

As these rewritings show, when this result for D_4 is substituted into the above integral equation for $F^{(2)^\dagger}$ the subtracted diagrams may either be used to modify the $\pi - N$ t-matrix

or the $N-N$ t-matrix. As will be seen below, which t-matrix we choose to modify here will affect our later results for other amplitudes. In this work we choose the former approach, however we stress that modifying the $N-N$ t-matrix is equally legitimate. When these subtractions are made we find that:

$$F^{(2)^\dagger} = \sum_{\alpha i} v^R(\alpha) d_\alpha^{-1} \bar{\delta}_{\alpha i} d_{\bar{i}} d_\pi f^{(1)^\dagger}(i) + \sum_\alpha t^{(2)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi F^{(2)^\dagger} \tag{58}$$

with:

$$v^R(\alpha) = \begin{cases} t_{\pi N}^{(2)}(i) - v^X(i) & \text{if } \alpha = i = 1,2 \\ T_{NN}^{(2)} & \text{if } \alpha = 3. \end{cases} \tag{59}$$

Note that:

1. If we choose to modify the NN t-matrix Eq. (58) still holds, but with $v^R(\alpha)$ given by:

$$v^R(\alpha) = \begin{cases} t_{\pi N}^{(2)}(i) & \text{if } \alpha = i = 1,2 \\ T_{NN}^{(2)} - v_{OPE} & \text{if } \alpha = 3. \end{cases} \tag{60}$$

2. Provided we use amplitudes $T_{NN}^{(2)}$ and $t_{\pi N}^{(2)}$ which themselves contain no double-counting the integral equation for $F^{(2)^\dagger}$, Eq. (58), is completely correct and, in turn, contains no double-counting.

3. There is a bootstrap problem here. We set out to determine $T_{NN}^{(2)}$ and found it depended on $F^{(2)^\dagger}$. Now we find that $F^{(2)^\dagger}$ depends on $T_{NN}^{(2)}$!

However, leaving this bootstrap problem aside for the moment, Eq. (58) may be iterated in order to sum the multiple-scattering series for $F^{(2)^\dagger}$. If this is done and the definitions (39)–(65) used, we obtain:

$$\begin{aligned} F^{(2)^\dagger} = & \sum_{\beta i} \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_1 d_2 d_\pi \bar{\delta}_{\beta i} f^{(1)^\dagger}(i) d_i^{-1} \\ & + \sum_{\alpha\beta i} t^{(1)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U_{\alpha\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_1 d_2 d_\pi \bar{\delta}_{\beta i} f^{(1)^\dagger}(i) d_i^{-1}, \end{aligned} \tag{61}$$

where the $U_{\alpha\beta}^{(2)}$s obey the four-dimensional covariant AGS equations [31]:

$$U_{\alpha\beta}^{(2)} = \bar{\delta}_{\alpha\beta} d_1^{-1} d_2^{-1} d_\pi^{-1} + \sum_\gamma \bar{\delta}_{\alpha\gamma} t^{(1)}(\gamma) d_\gamma^{-1} d_1 d_2 d_\pi U_{\gamma\beta}^{(2)}, \tag{62}$$

$$= \bar{\delta}_{\alpha\beta} d_1^{-1} d_2^{-1} d_\pi^{-1} + \sum_\gamma U_{\alpha\gamma}^{(2)} d_1 d_2 d_\pi t^{(1)}(\gamma) d_\gamma^{-1} \bar{\delta}_{\gamma\beta}, \tag{63}$$

and:

$$\tilde{t}_{\pi N}^{(1)} \equiv (1 + t_{\pi N}^{(1)} d_N d_\pi) v_{\pi N}^R = v_{\pi N}^R + t_{\pi N}^{(2)} d_N d_\pi \tilde{t}_{\pi N}^{(1)}, \tag{64}$$

$$\tilde{t}_{NN}^{(1)} \equiv T_{NN}^{(1)}, \tag{65}$$

where $T_{NN}^{(1)}$ is, in principle, the full NN t-matrix, which is related to $T_{NN}^{(2)}$ by Eq. (11).

Note that all two-body amplitudes in this series are the full s-channel 1PI two-body amplitudes $t^{(1)}$, except that the first $\pi - N$ scattering after the pion emission involves a t-matrix, $\tilde{t}_{\pi N}^{(1)}$, which has had the crossed term removed from the potential. This removal of the crossed term is the only way in which this result differs from that obtained by Avishai and Mizutani and Afnan and Blankleider. It is necessary because, as was first pointed out by Kowalski et al. [30], the inclusion of the crossed diagram in the first $\pi - N$ amplitude after pion emission leads to double-counting of terms such as Eq. (46) and (49) if the calculation is done in a time-dependent perturbation theory. Because we have used the modified Taylor method, which is specifically designed *not* to lead to double-counting, the above equation does not contain such double-counting, even though it was derived in a time-dependent perturbation theory.

VI. THE $N - N$ AMPLITUDE, PART 2

Having obtained the correct expression for $F^{(2)^\dagger}$ we may now return to our calculation of $T_{NN}^{(2)}$. Recall that above we derived Eq. (26) for $T_{NN}^{(2)}$, with the first term generated by C_5 and the second by C_4. However, recall also that both terms were expected to contain double-counting. Having derived an expression for $F^{(2)^\dagger}$ which is itself free of double-counting we may now determine the factors D_4 and D_5, which were introduced into Eq. (26) in order to compensate for this double-counting.

A. Calculating D_4

Firstly, note that:

$$D_4 = C_4^{\{N1'\}\{N2\}} \cap C_4^{\{N2'\}\{N1\}}, \tag{66}$$

with $C_4^{\{N1'\}\{N2\}}$ and $C_4^{\{N2'\}\{N1\}}$ given by the expressions (18) and (19).

Now observe that since all particles are fully dressed all vertices are at least 1PI in all channels. Therefore in a time-dependent perturbation theory:

$$\tilde{f}^{(1)\dagger} = f^{(2)}. \tag{67}$$

Consequently:

$$f^{(2)}(2) d_\pi \tilde{f}^{(1)\dagger}(1) = f^{(2)}(1) d_\pi \tilde{f}^{(1)\dagger}(2), \tag{68}$$

i.e. one sub-class merely reproduces the sum of the other. Topologically this happens because the two cuts $N1'\pi_I N2$ and $N2'\pi_I N1$ (here I stands for intermediate line) may be made on *any* diagram in C_4 and so all diagrams in C_4 belong to both sub-classes. Therefore:

$$D_4 = C_4^{\{N1'\}\{N2\}} = f^{(2)}(2) d_\pi \tilde{f}^{(1)\dagger}(1) = C_4^{\{N2\}\{N1'\}} = f^{(2)}(1) d_\pi \tilde{f}^{(1)\dagger}(2). \tag{69}$$

B. Calculating D_5

In Section IV we defined D_5 to be:

$$D_5 = C_5^{\{N1'\}} \cap C_5^{\{N2'\}}. \tag{70}$$

In order to discover which diagrams are actually in D_5 we examine the sum of $C_5^{\{N1'\}}$:

$$C_5^{\{N1'\}} = f^{(2)}(1) d_2 d_\pi {F^{(2)}}^\dagger, \tag{71}$$

and note which diagrams in $C_5^{\{N1'\}}$ admit a three-cut involving $N2'$ and so should be included in D_5. Naturally, provided that we use an expression for ${F^{(2)}}^\dagger$ which is itself free of double-counting, such as Eq. (61) derived above, the sum of $C_5^{\{N1'\}}$ will itself be free of double-counting. Consequently, such a procedure will correctly identify all of the diagrams which must be included for subtraction in D_5.

Now substituting in (71) for ${F^{(2)}}^\dagger$ from Eq. (61) and iterating using the AGS equations gives:

$$\begin{aligned} C_5^{\{N1'\}} =& \sum_{\alpha i} f^{(1)}(1) \bar{\delta}_{1\alpha} d_2 d_\pi \tilde{t}^{(1)}(\alpha) d_\alpha^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\alpha i} {f^{(1)}}^\dagger(i) + f^{(2)}(1) d_2 d_\pi \tilde{t}_{\pi N}^{(1)}(1) d_\pi {f^{(1)}}^\dagger(2) \\ + & \sum_{\alpha\beta i} f^{(1)}(1) \bar{\delta}_{1\alpha} d_2 d_\pi t^{(1)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U_{\alpha\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\beta i} {f^{(1)}}^\dagger(i). \end{aligned} \tag{72}$$

We write:

$$C_5^{\{N1'\}} = c_1 + c_2 + c_2 + c_3 + c_4 + c_5 + c_6, \tag{73}$$

where:

$$c_1 = f^{(1)}(1) d_\pi \tilde{t}_{\pi N}^{(1)}(2) d_2 d_\pi {f^{(1)}}^\dagger(1), \tag{74}$$

$$c_2 = f^{(1)}(1) d_2 d_\pi T_{NN}^{(1)} d_2 {f^{(1)}}^\dagger(1), \tag{75}$$

$$c_3 = f^{(1)}(1) d_2 d_\pi T_{NN}^{(1)} d_1 {f^{(1)}}^\dagger(2), \tag{76}$$

$$c_4 = f^{(2)}(1) d_2 d_\pi \tilde{t}_{\pi N}^{(1)}(1) d_\pi {f^{(1)}}^\dagger(2), \tag{77}$$

$$c_5 = \sum_{\beta i} f^{(1)}(1) d_2 d_\pi t_{\pi N}^{(1)}(2) d_1 d_\pi U_{2\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i), \tag{78}$$

$$c_6 = \sum_{\beta i} f^{(1)}(1) d_2 d_\pi T_{NN}^{(1)} d_1 d_2 U_{3\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i). \tag{79}$$

It is clear that:

$$D_5 = \sum_{a=1}^{6} c_a \cap C_5^{\{N2'\}}. \tag{80}$$

Therefore in order to determine the value of D_5 it is necessary to examine the six terms c_1–c_6 individually, probing carefully in order to determine which diagrams from each term admit a three-cut involving $N2'$.

1. c_1

We wish to calculate:

$$c_1 \cap C_5^{\{N2'\}}. \tag{81}$$

In order to do this we observe that if the diagram representing c_1 is to admit an s-channel three-cut involving $N2'$ and two internal lines then the amplitude $\tilde{t}^{(1)}_{\pi N}$ will have to be two-particle reducible in the $N' \leftarrow \pi' + \pi + N$ channel (see Fig. 7). Therefore we investigate $\tilde{t}^{(1)}_{\pi N}$ in order to discover what portion of it is two-particle reducible in this channel. We begin by using Eq. (64) to write:

$$\tilde{t}^{(1)}_{\pi N} = v^R + v^X d_N d_\pi \tilde{t}^{(1)}_{\pi N} + v^R d_N d_\pi \tilde{t}^{(1)}_{\pi N}. \tag{82}$$

The second term here is 2PR in the channel $N' \leftarrow \pi' + \pi + N$. Therefore, substituting this term into Eq. (74) for c_1 it is found that the diagram:

$$f^{(1)}_{\pi'}(1) d_{\pi'} f^{(1)}_{\pi}(2) d_1 {f^{(1)}_{\pi'}}^\dagger(2) d_1 d_\pi \tilde{t}^{(1)}_{\pi N}(2) d_2 d_\pi {f^{(1)}_{\pi}}^\dagger(1) \tag{83}$$

is in $c_1 \cap C_5^{\{N2'\}}$.

Examining (82) it is now seen that the only other way in which $\tilde{t}^{(1)}_{\pi N}$ could be 2PR in the $N' \leftarrow \pi' + \pi + N$-channel is if v^R is 2PR in the same channel. Therefore we now examine the effect of placing the $N' \leftarrow \pi' + \pi + N$-channel 2PR part of v^R in the expression for c_1.

Claim 1 *The portions of the expression:*

$$f^{(1)}(1) d_\pi \tilde{v}^{(2)}(2)(1 + d_1 d_\pi \tilde{t}^{(1)}_{\pi N}(2)) d_2 d_\pi {f^{(1)}}^\dagger(1),$$

where $\tilde{v}^{(2)}$ is the u-channel 1PI, s-channel 2PI and $N' \leftarrow \pi' + \pi + N$-channel 2PR $\pi - N$ interaction, which are also in $C_5^{\{N2'\}}$, given the assumptions of this calculation, are:

$$f^{(1)}_{\pi_2}(1) d_{\pi_2} f^{(1)}_{\pi_1}(2) d_1 t^{(1)}_{\pi_2 N}(2) d_{\pi_1} d_1 {f^{(1)}_{\pi_1}}^\dagger(2) d_{\pi_2} d_2 {f^{(1)}_{\pi_2}}^\dagger(1). \tag{84}$$

and:

$$\begin{aligned} &f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_{\pi_1} d_1 t^{(1)}_{\pi_1 \pi_2} d_{\pi_1} {f^{(1)}_{\pi_1}}^\dagger(2) d_{\pi_2} {f^{(1)}_{\pi_2}}^\dagger(1) \\ +&f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_{\pi_1} d_1 t^{(1)}_{\pi_1 \pi_2} d_{\pi_2} {f^{(1)}_{\pi_2}}^\dagger(2) d_{\pi_1} {f^{(1)}_{\pi_1}}^\dagger(1) \end{aligned} \tag{85}$$

For a proof of this result see Appendix C.

We never intended to include the possibility of $\pi - \pi$ interaction in our equations, since we think it is too small to make a significant difference to the calculation. This is a reasonable approximation, since in a model with only pions and nucleons two pions may only interact via the exchange of nucleon-anti-nucleon pairs. Consequently the lowest-order diagrams for $\pi - \pi$ scattering in a model with a πNN vertex and a $\pi - N$ contact term are the two diagrams shown in Figure 8. When these diagrams are used in (85) they produce a $N\bar{N}NN$ intermediate state. Considering that we are already neglecting explicit $\pi\pi NN$ states in

the calculation, and that we are also neglecting processes involving mesons other than the pion, e.g. the rho meson, the error made in ignoring diagrams such as those represented by Eq. (85) is insignificant, and consequently from now on we neglect the expression (85) as a possible source of double-counting.

Therefore the sum of all diagrams in $c_1 \cap C_5^{\{N2'\}}$ is:

$$\begin{aligned} & f_{\pi'}^{(1)}(1) d_{\pi'} f_{\pi}^{(1)}(2) d_1 f_{\pi'}^{(1)\dagger}(2) d_1 d_\pi \tilde{t}_{\pi N}^{(1)}(2) d_2 d_\pi f_{\pi}^{(1)\dagger}(1) \\ + & f_{\pi_2}^{(1)}(1) d_{\pi_2} f_{\pi_1}^{(1)}(2) d_1 t_{\pi_2 N}^{(1)}(2) d_1 d_{\pi_1} f_{\pi_1}^{(1)\dagger}(2) d_{\pi_2} d_2 f_{\pi_2}^{(1)\dagger}(1). \end{aligned} \tag{86}$$

2. c_2

The term c_2 has one possible cut involving $N2'$ and two internal lines, as shown in Figure 9. This diagram shows that any portion of $T_{NN}^{(1)}$ which is two-particle reducible in the $N1' \leftarrow N1 + N2 + N2'$-channel will, when placed in the expression for c_2, lead to diagrams which are in $c_2 \cap C_5^{\{N2'\}}$.

Consequently, we need to identify all $N1' \leftarrow N1 + N2 + N2'$-channel 2PR portions of $T_{NN}^{(1)}$. In order to do this write:

$$T_{NN}^{(1)} = T_{NN}^{(2)} + T_{NN}^{(2)} d_1 d_2 T_{NN}^{(1)}. \tag{87}$$

$T_{NN}^{(2)}$ is then split into two pieces: the t-channel 1PI part, which is denoted by $\tilde{T}_{NN}^{(2)}$, and the t-channel one-particle reducible part, which is clearly just the one-pion exchange potential. Therefore we obtain:

$$T_{NN}^{(2)} = v_{OPE} + \tilde{T}_{NN}^{(2)}, \tag{88}$$

(note that this is exactly the alternative decomposition given above in Eq. (60)) leading to:

$$T_{NN}^{(1)} = f^{(1)}(2) d_\pi f^{(1)\dagger}(1) + \tilde{T}_{NN}^{(2)} + f^{(1)}(2) d_\pi f^{(1)\dagger}(1) d_1 d_2 T_{NN}^{(1)} + \tilde{T}_{NN}^{(2)} d_1 d_2 T_{NN}^{(1)}. \tag{89}$$

Examination of the first term on the right-hand side of this equation shows that if $f^{(1)}(2)$ is two-particle reducible in the $N' \leftarrow N\pi'$ channel then this part of $T_{NN}^{(1)}$ will be 2PR in the $N1' \leftarrow N1 + N2 + N2'$-channel. Use of the LICL then shows that:

$$f_{\pi'}^{(1)}(1) d_2 d_{\pi'} f_{\pi}^{(2)}(2) d_1 d_\pi t_{\pi N}^{(1)}(2) d_\pi f_{\pi}^{(1)\dagger}(1) d_2 f_{\pi'}^{(1)\dagger}(1), \tag{90}$$

is in $c_2 \cap C_5^{\{N2'\}}$ and hence in D_5.

Furthermore, the third term on the right-hand side of (89) is completely 2PR in the $N1' \leftarrow N1 + N2 + N2'$-channel and thus the diagram:

$$f_{\pi}^{(1)}(1) d_2 d_\pi f_{\pi'}^{(1)}(2) d_{\pi'} f_{\pi'}^{(1)\dagger}(1) d_1 d_2 T_{NN}^{(1)} d_2 f_{\pi}^{(1)\dagger}(1), \tag{91}$$

is in the set $c_2 \cap C_5^{\{N2'\}}$.

Now consider the second and fourth terms of the right-hand side of Eq. (89). The only way in which these terms lead to portions of $T_{NN}^{(1)}$ which are 2PR in the $N1' \leftarrow N1+N2+N2'$-channel is if $\tilde{T}_{NN}^{(2)}$ is itself 2PR in the same channel. Arguments given in Appendix D show that the only contributions of such diagrams to $c_2 \cap C_5^{\{N2'\}}$ are:

$$f_{\pi_1}^{(1)}(2)d_{\pi_1}f_{\pi_2}^{(1)}(1)d_2t_{\pi_1N}^{(1)}(1)d_2d_{\pi_2}f_{\pi_2}^{(1)\dagger}(1)d_1d_{\pi_1}f_{\pi_1}^{(1)\dagger}(2), \tag{92}$$

and:

$$f_{\pi_1}^{(1)}(2)d_1d_{\pi_1}f_{\pi_2}^{(1)}(1)d_{\pi_2}d_2\tilde{T}_{NN}^{(1)}d_2f_{\pi_2}^{(1)\dagger}(1)d_1d_{\pi_1}f_{\pi_1}^{(1)\dagger}(2), \tag{93}$$

where $\tilde{T}_{NN}^{(1)}$ is not only one-particle irreducible in the s-channel, but also 1PI in the t-channel.

Therefore, the parts of c_2 which are also in $C_5^{\{N2'\}}$ and so are included in D_5 are the diagrams corresponding to Eqs. (90), (91), (92) and (93).

3. c_3

Examination of the diagram representing this term shows that two three-cuts involving $N2'$ are possible, as shown in Figure 10. The possibility of these three-cuts means that any diagram contributing to c_3 which contains a part of the NN t-matrix $T_{NN}^{(1)}$ which is:

1. 1PR in the t-channel;

2. 1PI in the t-channel but 2PR in the $N1' \leftarrow N1 + N2 + N2'$-channel,

is in $c_3 \cap C_5^{\{N2'\}}$.

As far as possibility 1 goes, substituting the 1PR piece of $T_{NN}^{(1)}$, v_{OPE}, into the expression (76) for c_3 indicates that the term:

$$f_{\pi_1}^{(1)}(1)d_2d_{\pi_1}f_{\pi_2}^{(1)}(2)d_{\pi_2}f_{\pi_2}^{(1)\dagger}(1)d_1f_{\pi_1}^{(1)\dagger}(2), \tag{94}$$

is in $c_3 \cap C_5^{\{N2'\}} \subset D_5$.

We also showed above and in Appendix D that the $N1' \leftarrow N1 + N2 + N2'$-channel 2PR parts of the t-channel 1PI NN t-matrix are:

1. $f^{(1)}(2)d_\pi f^{(1)\dagger}(1)d_1d_2T_{NN}^{(1)}$;

2. $f^{(1)}(2)d_1d_\pi\tilde{F}^{(2)\dagger}$;

3. $f^{(1)}(2)d_1d_\pi\tilde{F}^{(2)\dagger}d_1d_2T_{NN}^{(1)}$;

where the amplitude $\tilde{F}^{(2)\dagger}$ is defined in Appendix D. However, when these three diagrams are substituted into the expression for c_3 only diagram 1 and the portion:

$$f^{(1)}(2)d_1d_\pi T_{NN}^{(1)}d_2f^{(1)\dagger}(1) \tag{95}$$

of diagram 2 produce diagrams which are in $c_3 \cap C_5^{\{N2'\}}$. This is because all the other diagrams produced by the substitution of $N1' \leftarrow N1 + N2 + N2'$-channel 2PR parts of $\tilde{T}_{NN}^{(1)}$ into c_3 will only be contained in $C_5^{\{N2'\}}$ if a three-body force is included in the calculation.

Therefore the contents of $c_3 \cap C_5^{\{N2'\}}$ are the diagram (94), and the diagrams

$$f_\pi^{(1)}(1) d_\pi d_2 f_{\pi'}^{(1)}(2) d_{\pi'} f_{\pi'}^{(1)\dagger}(1) d_1 d_2 T_{NN}^{(1)} d_1 f_\pi^{(1)\dagger}(2) \tag{96}$$

and:

$$f_{\pi_1}^{(1)}(1) d_2 f_{\pi_2}^{(1)}(2) d_1 d_{\pi_1} T_{NN}^{(1)} d_{\pi_2} d_2 f_{\pi_2}^{(1)\dagger}(1) d_1 f_{\pi_1}^{(1)\dagger}(2). \tag{97}$$

4. c_4

If the three-cut drawn in Figure 11 may be made then the diagram representing c_4 will also be in $C_5^{\{N2'\}}$.

Therefore diagrams which sum to:

$$f^{(2)}(1) d_2 d_\pi \tilde{t}_{\pi N}^{(1)}(1) d_\pi f_{2PR}^{(1)\dagger}(2), \tag{98}$$

where $f_{2PR}^{(1)\dagger}(2)$ is the $N' \leftarrow N + \pi$-channel 2PR part of the 1PI πNN vertex, are in $c_4 \cap C_5^{\{N2'\}}$ and so are in D_5.

5. c_5

Examination of the diagram representing the sum of c_5 shows that an s-channel three-cut involving $N2'$ cannot be produced by cutting lines beyond the first interaction (see Fig. 12). Therefore, the structure of $U_{2\beta}^{(2)}$ and the amplitudes appearing after $U_{2\beta}^{(2)}$ are irrelevant to the calculation of $c_5 \cap C_5^{\{N2'\}}$.

Now an s-channel three-cut involving $N2'$ will be possible on this diagram if:

1. The $\pi - N$ t-matrix $t_{\pi N}^{(1)}$ used immediately before the pion absorption is u-channel 1PR;
2. The u-channel 1PI part of that $t_{\pi N}^{(1)}$ is $N' \leftarrow N + \pi + \pi'$-channel 2PR.

Using the above results for the u-channel 1PR part of $t_{\pi N}^{(1)}$ and the $N' \leftarrow N + \pi + \pi'$-channel 2PR part of the u-channel 1PI $t_{\pi N}^{(1)}$, but ignoring those diagrams which are only in $C_5^{\{N2'\}}$ if a three-body force is included in the calculation, shows that:

$$\begin{aligned} c_5 \cap C_5^{\{N2'\}} = \sum_{\beta i} & f_{\pi'}^{(1)}(1) d_2 d_{\pi'} f_\pi^{(1)}(2) d_1 f_{\pi'}^{(1)\dagger}(2)(1 + d_1 d_\pi t_{\pi N}^{(1)}(2)) \\ & \times d_1 d_\pi U_{2\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi f^{(1)\dagger}(i). \end{aligned} \tag{99}$$

6. c_6

As was the case for c_5, we observe (see Fig. 13) that an s-channel three-cut involving $N2'$ cannot be produced by cutting lines beyond the first interaction. Therefore, the structure of $U_{3\beta}^{(2)}$ and the amplitudes appearing after $U_{3\beta}^{(2)}$ are irrelevant to the calculation of $c_6 \cap C_5^{\{N2'\}}$.

Then an s-channel three-cut involving $N2'$ is possible in Fig. 13 if:

1. The 1PI $N-N$ t-matrix $T_{NN}^{(1)}$ used immediately before the pion absorption is t-channel 1PR.

2. The t-channel 1PI NN t-matrix, $\tilde{T}_{NN}^{(1)}$ is $N1' \leftarrow N1 + N2 + N2'$-channel 2PR.

Using the previous results for these portions of the NN t-matrix and again ignoring those parts of c_6 which are only also in $C_5^{\{N2'\}}$ if a three-body force is included in the calculation we find:

$$c_6 \cap C_5^{\{N2'\}} = \sum_{\beta i} f_\pi^{(1)}(1) d_2 d_\pi f_{\pi'}^{(1)}(2) d_{\pi'} {f_{\pi'}^{(1)}}^\dagger(1)(1 + d_1 d_2 T_{NN}^{(1)}) \\ \times d_1 d_2 U_{3\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i). \tag{100}$$

7. Overall result for D_5

Combining the results for the intersections $c_a \cap C_5^{\{N2'\}}$, $a = 1, \ldots, 6$, as per Eq. (80) reveals that:

$$\begin{aligned} D_5 = {} & \sum_{\beta i} f_{\pi'}^{(1)}(1) d_2 d_{\pi'} f_\pi^{(1)}(2) d_1 {f_{\pi'}^{(1)}}^\dagger(2)(1 + d_1 d_\pi t_{\pi N}^{(1)}(2)) d_1 d_\pi U_{2\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i) \\ & + \sum_{\beta i} f_\pi^{(1)}(1) d_2 d_\pi f_{\pi'}^{(1)}(2) d_{\pi'} {f_{\pi'}^{(1)}}^\dagger(1)(1 + d_1 d_2 T_{NN}^{(1)}) d_1 d_2 U_{3\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi {f^{(1)}}^\dagger(i) \\ & + f^{(2)}(1) d_2 d_\pi \tilde{t}_{\pi N}^{(1)}(1) d_\pi {f_{2PR}^{(1)}}^\dagger(2) \\ & + f_{\pi'}^{(1)}(1) d_2 d_{\pi'} f_\pi^{(2)}(2) d_1 d_\pi t_{\pi N}^{(1)}(2) d_\pi {f_\pi^{(1)}}^\dagger(1) d_2 {f_{\pi'}^{(1)}}^\dagger(1) \\ & + B + X + D_1 + D_2 + \tilde{D}_\pi \\ & + f_\pi^{(1)}(1) d_\pi d_2 f_{\pi'}^{(1)}(2) d_{\pi'} {f_{\pi'}^{(1)}}^\dagger(1) d_1 d_2 T_{NN}^{(1)} d_1 {f_\pi^{(1)}}^\dagger(2) \\ & + f_\pi^{(1)}(1) d_\pi d_2 f_{\pi'}^{(1)}(2) d_{\pi'} {f_{\pi'}^{(1)}}^\dagger(1) d_1 d_2 T_{NN}^{(1)} d_1 {f_\pi^{(1)}}^\dagger(1) \\ & + f_{\pi'}^{(1)}(1) d_{\pi'} f_\pi^{(1)}(2) d_1 {f_{\pi'}^{(1)}}^\dagger(2) d_1 d_\pi \tilde{t}_{\pi N}^{(1)}(2) d_2 d_\pi {f^{(1)}}^\dagger(1), \end{aligned} \tag{101}$$

where:

$$B = f_{\pi_1}^{(1)}(1) d_2 f_{\pi_2}^{(1)}(2) d_{\pi_1} d_{\pi_2} {f_{\pi_2}^{(1)}}^\dagger(1) d_1 {f_{\pi_1}^{(1)}}^\dagger(2), \tag{102}$$

represents crossed two-pion exchange; X is given by:

$$X = f_{\pi_1}^{(1)}(1) d_2 f_{\pi_2}^{(1)}(2) d_1 d_{\pi_1} T_{NN}^{(1)} d_{\pi_2} d_2 {f_{\pi_2}^{(1)}}^\dagger(1) d_1 {f_{\pi_1}^{(1)}}^\dagger(2), \tag{103}$$

the terms D_i are:

$$D_i = f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_1 d_{\pi_1} t^{(1)}_{\pi_i N}(i) d_i^{-1} d_1 d_{\pi_i} {f^{(1)}_{\pi_1}}^\dagger(2) d_2 {f^{(1)}_{\pi_2}}^\dagger(1), \tag{104}$$

and $\tilde{D}_\pi$ is given by:

$$\tilde{D}_\pi = f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_1 d_{\pi_1} \tilde{T}^{(1)}_{NN} d_1 {f^{(1)}_{\pi_1}}^\dagger(2) d_2 {f^{(1)}_{\pi_2}}^\dagger(1). \tag{105}$$

At this stage we must make a decision. Upon substitution of the expression for D_5 into (26) many of the terms in D_5 can be subtracted from either the $N-N$ t-matrix or the $\pi - N$ t-matrix. In what follows we choose to subtract them from the $\pi - N$ t-matrix, as we did above for the t-matrices in ${F^{(2)}}^\dagger$. This has the effect that many of the "potentials" generating the $\pi - N$ t-matrices, which are, by definition, 2PI in the s-channel, also become 1PI in the u-channel. In other words, the resultant input πN t-matrix appearing at some points in the equations has the crossed diagram removed from its "potential". However, we stress that it would be equally valid to modify the NN t-matrix. Were this to be done the modified NN "potential" would be 2PI in the s-channel *and* 1PI in the t-channel. That is, if we chose to proceed in this way the input NN t-matrix which appears at some points in the equations will not contain the one-pion exchange contribution in the driving term of its Bethe-Salpeter equation.

As we shall see further below, the fact that we choose to modify the $\pi - N$ t-matrix means that the πNN form factor in the one-pion exchange part of $T^{(2)}_{NN}$ needs to be adjusted. By contrast, if we chose to modify the NN t-matrix it may be shown that this change to the πNN form factor in one-pion exchange is unnecessary. This result is merely a specific example of the wider truth that any adjustments which are made in the pion production/absorption sector have significant implications for the $NN \to NN$ sector of the theory.

C. Equation for $T^{(2)}_{NN}$

Having made this decision we may substitute eqs.(61), (69) and (101) into Eq. (26), and simplify using the AGS equations in order to obtain:

$$\begin{aligned} T^{(2)}_{NN} &= f^{(2)}(1) d_\pi \tilde{f}^{(1)\dagger}(2) + \tilde{C}^d_5 \\ &+ \sum_{j\alpha i} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{\tilde{t}}^{(1)}(\alpha) d_\alpha^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\alpha i} {f^{(1)}}^\dagger(i) \\ &+ \sum_{j\alpha\beta i} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U^{(2)}_{\alpha\beta} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\beta i} {f^{(1)}}^\dagger(i) \\ &- D_1 - D_2 - \tilde{D}_\pi - X - B, \end{aligned} \tag{106}$$

where:

$$\tilde{t}^{(1)\dagger}_{\pi N} = v^R + v^R d_N d_\pi t^{(1)}, \tag{107}$$

and:

$$\tilde{t}^{(1)}_{\pi N} = v^R + t^{(1)} d_N d_\pi v^R, \tag{108}$$

are both one-particle irreducible in the s and u-channels, but differ in that $\tilde{t}^{(1)}_{\pi N}$ has the crossed term subtracted from the first interaction, while $\tilde{t}^{(1)\dagger}_{\pi N}$ has it subtracted from the last interaction. By contrast:

$$\tilde{\tilde{t}}^{(1)}(i) = \tilde{\tilde{t}}^{(1)}_{\pi N} = v^R + v^R d_N d_\pi v^R + v^R d_N d_\pi t^{(1)}_{\pi N} d_N d_\pi v^R, \tag{109}$$

for $i = 1, 2$, has the crossed term subtracted from both the first and the last term. Meanwhile:

$$\tilde{\tilde{t}}^{(1)}(3) = T^{(1)}_{NN}. \tag{110}$$

and $\tilde{C}^d_5$ is that portion of the sum of class C_5 which contributes to the dressing of one-pion exchange and is given by:

$$\begin{aligned}
\tilde{C}^d_5 = {} & f^{(2)}(2) d_1 d_\pi \tilde{t}^{(1)}_{\pi N}(2) d_\pi {f^{(1)}}^\dagger(1) \\
& + f^{(2)}(1) d_1 d_\pi \tilde{t}^{(1)}_{\pi N}(1) d_\pi \tilde{f}^{(1)\dagger}(2) \\
& - f^{(2)}_\pi(2) d_1 d_\pi t^{(1)}_{\pi N}(2) d_\pi f^{(1)}_{\pi'}(1) d_2 {f^{(1)}_\pi}^\dagger(1) d_2 d_{\pi'} {f^{(1)}_{\pi'}}^\dagger(1).
\end{aligned} \tag{111}$$

Formal manipulations now show that:

$$\tilde{C}^d_5 + f^{(2)}(1) d_\pi \tilde{f}^{(1)\dagger}(2) = V^*_{OPE} + D_{OPE}, \tag{112}$$

where:

$$D_{OPE} = f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_1 d_{\pi_1} f^{(1)}_{\pi_3}(2) d_{\pi_3} {f^{(1)}_{\pi_3}}^\dagger(1) d_1 {f^{(1)}_{\pi_1}}^\dagger(2) d_2 {f^{(1)}_{\pi_2}}^\dagger(1); \tag{113}$$

$$V^*_{OPE} = f^{(1)*}(1) d_\pi {f^{(1)*}}^\dagger(2); \tag{114}$$

with:

$$f^{(1)*} = f^{(2)} + f^{(2)} d_N d_\pi \tilde{t}^{(1)}_{\pi N}, \tag{115}$$

the vertex which is obtained if the crossed diagram is eliminated (in the manner prescribed above) from the $\pi - N$ t-matrix used to generate the dressed πNN vertex. Note that, using the definition of $\tilde{t}_{\pi N}$, Eq. (64), we find:

$$f^{(1)*} = f^{(1)} - f^{(1)}_{\pi'} d_N f^{(1)}_\pi d_N d_{\pi'} {f^{(1)}_{\pi'}}^\dagger \tag{116}$$

as depicted in Figure 14. Note that in order to obtain the result (112) we have had to use the time-dependent perturbation theory identities (52) and (67).

Resubstituting the result (112) into Eq. (106) yields the final result for $T^{(2)}_{NN}$:

$$\begin{aligned}
T^{(2)}_{NN} = {} & V^*_{OPE} - D_1 - D_2 - D_\pi - X - B \\
& + \sum_{j\alpha i} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{\tilde{t}}^{(1)}(\alpha) d_\alpha^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\alpha i} {f^{(1)}}^\dagger(i) \\
& + \sum_{j\alpha\beta i} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U^{(2)}_{\alpha\beta} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\beta i} {f^{(1)}}^\dagger(i),
\end{aligned} \tag{117}$$

where:

$$D_\pi = \tilde{D}_\pi + D_{OPE} = f^{(1)}_{\pi_2}(1) d_{\pi_2} d_2 f^{(1)}_{\pi_1}(2) d_{\pi_1} d_1 T^{(1)}_{NN} d_1 {f^{(1)}_{\pi_1}}^\dagger(2) d_2 {f^{(1)}_{\pi_2}}^\dagger(1). \tag{118}$$

The amplitude $T^{(2)}_{NN}$ is the nucleon-nucleon s-channel two-particle irreducible interaction which is to be used in the Bethe-Salpeter equation for NN scattering in this theory, Eq. (11). Hence it plays the role of the nucleon-nucleon "potential" in this work. However, rather than merely being the sum of a few Feynman diagrams $T^{(2)}_{NN}$ is the sum of *all* 2PI Feynman diagrams with one explicit pion. Consequently it includes the full explicit coupling to the πNN channel.

If this result is compared to the equation for $T^{(2)}_{NN}$ obtained in the derivation of the standard $NN - \pi NN$ equations [2–12] a number of differences are seen:

1. The one-pion exchange part of the "potential" has been replaced by the modified "potential":

$$\bar{V} = V^*_{OPE} - D_1 - D_2 - D_\pi - X - B, \tag{119}$$

   that actually contains (for reasons to be discussed below) the diagrams D_1, D_2, D_π, X and B which involve multiple pion exchanges.

2. In the two terms describing pion rescattering (i.e. the last two terms on the right-hand side of Eq. (117)) the $\pi - N$ t-matrix has been modified. In the first term the usual 1PI $\pi - N$ t-matrix has been supplanted by the t-matrix $\tilde{\tilde{t}}^{(1)}$, in which the crossed diagram has been removed from both the first and the last pion-nucleon "potential". Likewise, in the second term the operators $\tilde{t}^{(1)}$ and $\tilde{t}^{(1)\dagger}$ govern, respectively, the first/last pion rescattering after/before pion emission/absorption.

All of these changes are the result of our desire to eliminate double-counting from the equations. Taking the second set of changes first, in Section V it was shown that either the πN or NN amplitude immediately following a pion emission needed to be modified in order to avoid the over-counting of diagrams involving the process (55). The changes to these $\pi - N$ t-matrices in Eq. (117) are made for exactly the same reason. The effect is to place constraints on these sub-amplitudes in channels other than the s-channel. As discussed in Section III, such additional constraints are necessary because in a time-dependent perturbation theory diagram constraints on the s-channel cut-structure of sub-amplitudes are not enough to constrain the overall s-channel cut-structure of the amplitude.

Turning now to the first set of changes, we divide our discussion into two halves. Firstly, let us consider the one-pion exchange piece of $\bar{V}$, V^*_{OPE}. A very interesting consequence of our work here is that the one-pion exchange "potential" V^*_{OPE} contains a different vertex than that used in the pole part of the $\pi - N$ t-matrix $t^{(0)}_{\pi N}$. This is necessary because the inclusion of the full πNN vertex in the one-pion exchange potential leads to double-counting with respect to other terms in $T^{(2)}_{NN}$. (See Fig. 15 for examples of such diagrams.) These other terms are included in the final two terms of Eq. (117). Therefore, they only arise when the $NN\pi$ cut structure of the amplitude $T^{(2)}_{NN}$ is considered. This is a manifestation of the fact that the amount of double-counting to be eliminated is entirely dependent upon

which unitarity cuts are opened in the analysis. Our above discovery that certain types of double-counting do not arise when a three-body force is excluded from the calculation is another manifestation of the same effect. Therefore we find that whether or not the vertex used for the one-pion exchange piece of $T_{NN}^{(2)}$ needs to be adjusted depends on whether:

1. the πNN unitarity cut is opened or not;
2. the subtraction terms are included in $t_{\pi N}$ or T_{NN} (see discussion in Section VI B).

Secondly, the other subtractions in $\bar{V}$ (i.e. D_1, D_2, D_π, B and X) are also required in order to correct for the overcounting of diagrams in the last two terms of Eq. (117). (Note that once again it is the opening of the πNN cut which has led to double-counting.) As is shown in Figs. 16 and 17 D_1, D_2, D_π, B and X are all included in this term in more than one place. Consequently these five diagrams will be double-counted unless some adjustment to the equation for $T_{NN}^{(2)}$ is made. Because we wish to obtain a set of coupled equations we choose to make this adjustment in the NN "potential", rather than in the pion absorption/production channel.

The necessity of eliminating double-counting from the theory is therefore forcing either:

1. The modification of the amplitudes appearing within the equations, or
2. The inclusion of explicit subtraction terms to compensate for the over-counting.

If option 1 is pursued the result is that different amplitudes appear at different places within the theory: for instance, we have four πN amplitudes in the equation (117). If option 2 is taken then the presence of extra subtraction terms means that the driving term of the equation takes on a far more complicated form, since it now involves diagrams with more than one explicit pion. In either case the original goal of deriving coupled equations for the $NN - \pi NN$ system which contain only one explicit pion and standard 1PI NN and πN amplitudes has been defeated. If such equations were to be derived, as was done by AM and AB, certain diagrams would need to be (incorrectly) included in *two* (or more) places in the theory. That these diagrams appear *once and only once* in the diagrammatic expansion is a direct consequence of the indeterminacy of the time-ordering of interaction vertices on different particles in a time-dependent field theory.

VII. DERIVATION OF COUPLED EQUATIONS FOR THE $NN - \pi NN$ SYSTEM

Our next task is to obtain double-counting-free coupled integral equations for the amplitudes which describe the processes:

$$\left.\begin{array}{r} N_1 + N_2 \\ (N_2\pi) + N_1 \\ (N_1\pi) + N_2 \\ (N_1N_2) + \pi \end{array}\right\} \longrightarrow \left\{\begin{array}{l} N_1 + N_2 \\ (N_2\pi) + N_1 \\ (N_1\pi) + N_2 \\ (N_1N_2) + \pi \end{array}\right. \tag{120}$$

This will be achieved in two stages: firstly, integral equations governing the amplitudes which describe the reactions $NN \to NN$, $NN \to NN\pi$, $NN\pi \to NN$, $NN\pi \to NN\pi$ will be found. Once this is accomplished the residues of these amplitudes at the appropriate poles will be taken, in order to derive equations connecting the two-fragment amplitudes for the above processes.

A. Deriving integral equations for the two and three-body amplitudes

In this section the first of these two steps is performed: the derivation of integral equations for the two and three-body amplitudes. We have, of course, already begun this task by deriving equations for the amplitudes $T_{NN}^{(1)}$, $T_{NN}^{(2)}$ and $F^{(2)\dagger}$. But, equations are now needed, not only for these amplitudes, but also for the connected $NN\pi \to NN\pi$ amplitude $M^{(1)}$, the connected $NN \to NN\pi$ amplitude $F^{(1)\dagger}$, and the connected $NN\pi \to NN$ amplitude, $F^{(1)}$. (Note that since all the amplitudes under consideration involve a two-nucleon initial state, by conservation of nucleon number they are automatically one-particle irreducible.)

1. $F^{(1)\dagger}$ and $F^{(1)}$

Firstly, consider the connected 1PI $NN \to NN\pi$ amplitude $F^{(1)\dagger}$. Since the parameters of the Taylor method for this case are $n = 3$, $m = 2$ and $r = 2$ no double-counting can arise and so the classification-of-diagrams technique may be applied to this amplitude without adaptation. This process produces the equation:

$$F^{(1)\dagger} = F^{(2)\dagger}(1 + d_1 d_2 T_{NN}^{(1)}) + \sum_j f^{(1)\dagger}(j) d_1 d_2 T_{NN}^{(1)}, \tag{121}$$

where the first term is generated by C_1, the second by C_2, and the third by C_5. Classes C_3 and C_4 are empty, since, as has already been observed above, all one-to-one amplitudes are zero.

Substituting the expression obtained for $F^{(2)\dagger}$ given in Eq. (61) then produces:

$$F^{(1)\dagger} = \sum_j f^{(1)\dagger}(j) d_1 d_2 T_{NN}^{(1)} + \left[\sum_{\alpha i} \tilde{t}^{(1)}(\alpha) d_\alpha^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\alpha i} f^{(1)\dagger}(i) \right.$$
$$\left. + \sum_{\alpha\beta i} t^{(1)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U_{\alpha\beta}^{(2)} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi f^{(1)\dagger}(i)\right] (1 + d_1 d_2 T_{NN}^{(1)}). \tag{122}$$

Taking the adjoint of Eq. (122) then gives an expression for the 1PI $NN\pi \to NN$ amplitude, $F^{(1)}$, in terms of two-body t-matrices and dressed πNN vertices.

2. $M^{(1)}$

In this case $m = n = 3$ and $r = 2$, and therefore Taylor's original method may again be applied, giving:

$$M^{(1)} = M^{(2)} + F^{(2)\dagger} d_1 d_2 F^{(1)} + F^{(2)\dagger} d_1 d_2 \left[\sum_{i=1}^{2} f^{(1)}(i) d_i^{-1}\right] + \left[\sum_{j=1}^{2} f^{(1)\dagger}(j) d_j^{-1}\right] d_1 d_2 F^{(1)}, \tag{123}$$

where the four terms come from, respectively, C_1, C_2, C_3 and C_5. Note that C_4 is empty. Note also that we have made use of Eq. (B6) for $f^{(0)}$ here.

In order to derive an expression for $M^{(1)}$ in terms of two-body t-matrices and dressed πNN vertices an equation for $M^{(2)}$ must be derived. In the case of $M^{(2)}$ the parameters

of the Taylor method are $n = m = r = 3$ and so one might expect that, since $n = r$, double-counting will occur in C_4 and C_5. However, since all of the particles are fully dressed no three-cut may intersect more than one line from the final state and so in every diagram in C_4 and C_5 we have $s_f = 1$. Consequently, the condition, $n > 2s_f$ is always satisfied and so it is guaranteed that no double-counting occurs even if Taylor's original method is used. When the original Taylor method is applied to the amplitude $M^{(2)}$ it yields:

$$M^{(2)} = M^{(3)} + \left\{ \left[M^{(3)} + \sum_{\alpha} t^{(2)}(\alpha) d_{\alpha}^{-1} \right] d_1 d_2 d_\pi \left[M^{(2)} + \sum_{\beta} t^{(1)}(\beta) d_{\beta}^{-1} \right] \right\}^{(c)}, \tag{124}$$

where the superscript $^{(c)}$ indicates that the amplitude resulting from the multiplication must be connected.

Since we have consistently ignored three-body $NN\pi$ forces throughout this calculation, we set $M^{(3)}$ to zero, thus producing the equation:

$$M^{(2)} = \sum_{\alpha\beta} t^{(2)}(\alpha) d_{\alpha}^{-1} d_1 d_2 d_\pi \bar{\delta}_{\alpha\beta} t^{(1)}(\beta) d_{\beta}^{-1} + \sum_{\alpha} t^{(2)}(\alpha) d_{\alpha}^{-1} d_1 d_2 d_\pi M^{(2)}. \tag{125}$$

This equation may then be formally solved, in order to yield:

$$M^{(2)} = \sum_{\alpha\beta} t^{(1)}(\alpha) d_{\alpha}^{-1} d_1 d_2 d_\pi U_{\alpha\beta}^{(2)} d_1 d_2 d_\pi t^{(1)}(\beta) d_{\beta}^{-1}, \tag{126}$$

where $U_{\alpha\beta}^{(2)}$ are the covariant AGS amplitudes, which obey Eqs. (62) and (63).

Using Eqs. (61), (122) and (126) in Eq.(123) an expression for $M^{(1)}$ in terms of the πN and NN amplitudes and the dressed πNN vertex may be derived.

B. Coupled equations for the $NN \to NN$ and $NN\pi \to NN$ reactions

Eqs. (11), (117), (122) and (123) are equations for the two and three-body amplitudes of interest. However, our goal is to derive equations connecting the amplitudes for the collision of two fragments, rather than equations containing the three-body amplitudes. So consider firstly the reactions in which a two-fragment final state that includes a pion is produced from an initial NN state:

$$N_1 + N_2 \longrightarrow \begin{cases} (N_2\pi) + N_1 \\ (N_1\pi) + N_2 \\ (N_1 N_2) + \pi \end{cases} \tag{127}$$

Suppose that the final state of interest is one in which the particle λ is a spectator while the other two particles are in a bound or resonance state. If the energy of this final state is E_λ then the matrix element for such a process, $X_{\lambda N}^{D}$, is given by [32]:

$$X_{\lambda N}^{D} = (-i) \langle \phi(\lambda) | \langle \chi_\lambda | d_{\lambda}^{-1} d_1 d_2 d_\pi \mathrm{Res}_{\lambda \text{ pole}} d_1 d_2 d_\pi d_{\lambda}^{-1} F_D^{(1)^\dagger} | \psi_D \rangle \tag{128}$$

where $|\phi(\lambda)\rangle$ is the form factor for the formation of the bound or resonance state; $|\chi_\lambda\rangle$ and d_λ are, respectively, the wave function and dressed propagator for a free λ particle and $|\psi_D\rangle$

is a wave function for the initial pair N_1 and N_2. Note that so far we have treated all particles as distinguishable, hence the initial state NN wave function must be the one for distinguishable particles, and the form-factor $|\phi(\lambda)\rangle$ should not be anti-symmetrized, even if λ represents the channel $(N_1N_2)+\pi$. Furthermore, we indicate the fact that the equations for $F^{(1)\dagger}$ assume distinguishable particles by placing the subscript $_D$ on $F^{(1)\dagger}$.

In order to derive an equation for $X^D_{\lambda N}$ the residue of Eq. (122) must be taken at the final state λ pole. It is therefore necessary to make explicit the analytic structure "near" this pole of each of the amplitudes which appear last in the various terms on the right-hand side of Eq. (122). To facilitate this we first write:

$$\tilde{t}^{(1)}(\alpha) = \begin{cases} t^{(1)}(\alpha) - t^{(1)}(\alpha)d_N d_\pi v^X(\alpha) - v^X(\alpha) & \text{if } \alpha = 1,2 \\ t^{(1)}(\alpha) & \text{if } \alpha = 3 \end{cases} \tag{129}$$

in Eq. (122). Now, if p_α is the total four-momentum of the two-body sub-system labeled by α then it may be shown that the analytic structure of $t^{(1)}(\alpha)$ "near" the α pole, $p_\alpha^2 = M_\alpha^2$, is given by:

$$t^{(1)}(\alpha) \overset{p_\alpha^2\to M_\alpha^2}{\sim} |\phi(\alpha)\rangle \frac{i}{p_\alpha^2 - M_\alpha^2}\langle\phi(\alpha)| + \text{(regular terms)}. \tag{130}$$

When the three-body operator $t^{(1)}(\alpha)d_\alpha^{-1}$ is considered it is found that if the four-momentum of the spectator α is k_α, the α pole is at the point $(P-k_\alpha)^2 = M_\alpha^2$ where P is the total four-momentum of the system. Near that point:

$$\begin{aligned} t^{(1)}(\alpha)d_\alpha^{-1}(k_\alpha) &\overset{(P-k_\alpha)^2\to M_\alpha^2}{\sim} d_\alpha^{-1}(k_\alpha)|\phi(\alpha)\rangle|\chi_\alpha(k_\alpha)\rangle \frac{i}{(P-k_\alpha)^2 - M_\alpha^2} d_\alpha(k_\alpha)\langle\phi(\alpha)|\langle\chi_\alpha(k_\alpha)|d_\alpha^{-1}(k_\alpha) \\ &+ \quad \text{(regular terms)}. \end{aligned} \tag{131}$$

The amplitudes other than $t^{(1)}$ which appear last in some term in Eq. (122) are v^X and $f^{(1)\dagger}$. Since the former is the crossed term of the $\pi - N$ potential and the latter is the dressed $N \to \pi N$ form factor, neither v^X or $f^{(1)\dagger}$ have a pole at any of the points $(P-k_\lambda)^2 = E_\lambda^2$. (The exception to this statement occurs if $t^{(1)}_{\pi N}$ includes a resonance with the same quantum numbers as the nucleon. In this case care must be exercised as the residue may receive a contribution from $f^{(1)\dagger} = (1 + t^{(1)}_{\pi N} d_N d_\pi) f^{(2)\dagger}$.) Consequently, any term in which a v^X or $f^{(1)\dagger}$ is the last amplitude disappears upon the taking of the residue. It follows that Eqs. (128) and (122) lead to the following form for $X^D_{\lambda N}$:

$$X^D_{\lambda N} = \langle\phi(\lambda)|\langle\chi_\lambda|d_\lambda^{-1} d_1 d_2 d_\pi T^D_{\lambda N}|\psi_D\rangle, \tag{132}$$

where the amplitude $T^D_{\lambda N}$ is given by:

$$\begin{aligned} T^D_{\lambda N} &= \sum_j (1 - v^X(\lambda)d_\lambda^{-1} d_1 d_2 d_\pi)\bar{\delta}_{\lambda j} f^{(1)\dagger}(j) d_j^{-1}(1 + d_1 d_2 T^D_{NN}) \\ &+ \sum_{\beta i} U^{(2)}_{\lambda\beta} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} \bar{\delta}_{\beta i} d_{\bar{i}} d_\pi f^{(1)\dagger}(i)(1 + d_1 d_2 T^D_{NN}). \end{aligned} \tag{133}$$

Here we have replaced the NN scattering amplitude $T^{(1)}_{NN}$ by the operator T^D_{NN}, since, as discussed above, the 1PI NN scattering amplitude is the full NN scattering amplitude. The AGS equations may now be used in order to obtain:

$$T^D_{\lambda N} = \left[\sum_j (1 - v^X(\lambda) d_\lambda^{-1} d_1 d_2 d_\pi) \bar{\delta}_{\lambda j} f^{(1)^\dagger}(j) d_j^{-1} - \sum_{\alpha j} \bar{\delta}_{\lambda\alpha} v^X(\alpha) d_\alpha^{-1} d_{\bar{j}} d_\pi \bar{\delta}_{\alpha j} f^{(1)^\dagger}(j)\right]$$
$$\times (1 + d_1 d_2 T^D_{NN}) + \sum_\alpha \bar{\delta}_{\lambda\alpha} t^{(1)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi T^D_{\alpha N}. \qquad (134)$$

Note that this equation is equivalent to that derived by Avishai and Mizutani (AM) [10] and Afnan and Blankleider (AB) [11], except for the presence of the term:

$$-\left[\sum_{\alpha j} v^X(\alpha) d_\alpha^{-1} d_{\bar{j}} d_\pi \bar{\delta}_{\alpha j} f^{(1)^\dagger}(j)\right](1 + d_1 d_2 T^D_{NN}), \qquad (135)$$

which has been included in order to remove the double-counting present when the equations derived by AM and AB are used in a time-dependent perturbation theory.

The equation derived for $T^D_{\lambda N}$ contains the amplitude T^D_{NN}. Therefore it is now necessary to derive an equation for T^D_{NN}. Recall that Eqs. (11) and (117) are the results given by the modified Taylor method for T_{NN}. Substituting (117) into (11) yields:

$$T^D_{NN} = (V^*_{OPE} - D_1 - D_2 - D_\pi - X - B)(1 + d_1 d_2 T^D_{NN}) + \left(\sum_{j\alpha i} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1}\right)$$
$$d_1 d_2 d_\pi \left[(1 - v^X(\alpha) d_\alpha^{-1}) d_1 d_2 d_\pi \bar{\delta}_{\alpha i} f^{(1)^\dagger}(i) d_i^{-1} + \sum_\beta U^{(2)}_{\alpha\beta} d_1 d_2 d_\pi \tilde{t}^{(1)}(\beta) d_\beta^{-1} d_{\bar{i}} d_\pi \bar{\delta}_{\beta i} f^{(1)^\dagger}(i)\right]$$
$$(1 + d_1 d_2 T^D_{NN}). \qquad (136)$$

Using Eq. (133) for $T_{\alpha N}$ then reveals that:

$$T^D_{NN} = \bar{V}(1 + d_1 d_2 T^D_{NN}) + \sum_{j\alpha} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi T^D_{\alpha N}, \qquad (137)$$

where $\bar{V}$ is given by Eq. (119).

Recall that in the discussion after Eq. (117) it was emphasized that the final result for $\bar{V}$ is dependent upon the treatment of the $NN \leftrightarrow NN\pi$ sector of the theory. Similarly, since Eq. (137) results from a consistent treatment of the coupled $NN - \pi NN$ system one cannot simple-mindedly modify this result by, for instance, merely dropping the second term on the right-hand side. Any such change must be developed in a coherent way throughout the theory, rather than imposed in an *ad hoc* way on the final equations.

Because we have used the modified Taylor method and so eliminated all double-counting the set of four equations given in Eqs. (134) and (137) are double-counting-free coupled four-dimensional integral equations for the $NN \to \pi NN$ and $NN \to NN$ channels.

C. Coupled equations for the $\pi NN \to \pi NN$ and $NN \to NN\pi$ reactions

We now turn our attention to the $\pi NN \to \pi NN$ and $NN \to \pi NN$ channels, and attempt to derive coupled equations for the reactions:

$$\left.\begin{array}{l}(N_2\pi)+N_1\\(N_1\pi)+N_2\\(N_1N_2)+\pi\end{array}\right\}\longrightarrow\left\{\begin{array}{l}(N_2\pi)+N_1\\(N_1\pi)+N_2\\(N_1N_2)+\pi\end{array}\right. \tag{138}$$

and:

$$N_1+N_2\longrightarrow\left\{\begin{array}{l}(N_2\pi)+N_1\\(N_1\pi)+N_2\\(N_1N_2)+\pi.\end{array}\right. \tag{139}$$

The two-to-two amplitude for any of the reactions (138) is found by taking the right and left residue of $M_D^{(1)}$ at the relevant poles. That is, the matrix element for a transition from a state in which particle λ is a spectator to a state in which particle μ is a spectator is $X^D_{\lambda\mu}$, given by:

$$\begin{aligned}X^D_{\lambda\mu}=(-i)^2\langle\phi(\lambda)|\langle\chi_\lambda|d_\lambda^{-1}d_1d_2d_\pi\mathrm{Res}_{\lambda\text{ pole}}d_1d_2d_\pi d_\lambda^{-1}M_D^{(1)}d_\mu^{-1}d_1d_2d_\pi\\ \mathrm{Res}_{\mu\text{ pole}}d_1d_2d_\pi d_\mu^{-1}|\phi(\mu)\rangle|\chi_\mu\rangle.\end{aligned} \tag{140}$$

In order to derive an equation for $T^D_{\lambda\mu}$, Eq. (123) is used to glean an expression for $M^{(1)}$ in terms of the input to the equations: $t_{\pi N}$, t_{NN} and $f^{(1)}$. Using the facts about the analytic structure of these amplitudes which were given above and taking right and left residues of this expression as per Eq. (140), it is found that:

$$X^D_{\lambda\mu}=\langle\phi(\lambda)|\langle\chi_\lambda|d_\lambda^{-1}d_1d_2d_\pi T^D_{\lambda\mu}d_1d_2d_\pi d_\mu^{-1}|\phi(\mu)\rangle|\chi_\mu\rangle, \tag{141}$$

with:

$$\begin{aligned}T^D_{\lambda\mu}=U^{(2)}_{\lambda\mu}+\Bigg[&\sum_j(1-v^X(\lambda)d_\lambda^{-1}d_1d_2d_\pi)\bar{\delta}_{\lambda j}{f^{(1)}}^\dagger(j)d_j^{-1}\\ &+\sum_{\delta j}U^{(2)}_{\lambda\delta}d_1d_2d_\pi\tilde{t}^{(1)}(\delta)d_\delta^{-1}d_{\bar{j}}d_\pi\bar{\delta}_{\delta j}{f^{(1)}}^\dagger(j)\Bigg]d_1d_2\\ &(T^D_{NN}d_1d_2+1)\Bigg[\sum_{i\gamma}f^{(1)}(i)\bar{\delta}_{i\gamma}d_{\bar{i}}d_\pi\tilde{t}^{(1)\dagger}(\gamma)d_\gamma^{-1}d_1d_2d_\pi U^{(2)}_{\gamma\mu}\\ &+\sum_i f^{(1)}(i)d_i^{-1}\bar{\delta}_{i\mu}(1-d_1d_2d_\pi v^X(\mu)d_\mu^{-1})\Bigg].\end{aligned} \tag{142}$$

We may also define $X^D_{N\mu}$ as the two-fragment matrix element for transition from the two-body state in which particle μ is a spectator to the state containing two (distinguishable) nucleons:

$$X^D_{N\mu}=(-i)\langle\psi_D|F_D^{(1)}d_\mu^{-1}d_1d_2d_\pi\mathrm{Res}_{\mu\text{ pole}}d_1d_2d_\pi d_\mu^{-1}|\phi(\mu)\rangle|\chi_\mu\rangle d_\mu^{-1}. \tag{143}$$

Once more it is found that:

$$X^D_{N\mu}=\langle\psi_D|T^D_{N\mu}d_1d_2d_\pi d_\mu^{-1}|\phi(\mu)\rangle|\chi_\mu\rangle, \tag{144}$$

this time with:

$$T^D_{N\mu} = (T^D_{NN} d_1 d_2 + 1) \left[\sum_{j\alpha} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi U^{(2)}_{\alpha\mu} \right.$$
$$\left. + \sum_j f^{(1)}(j) d_j^{-1} \bar{\delta}_{j\mu} (1 - d_1 d_2 d_\pi v^X(\mu) d_\mu^{-1}) \right]. \tag{145}$$

Using the covariant AGS equations and Eq. (145), Eq. (142) may now be simplified to:

$$T^D_{\lambda\mu} = \bar{\delta}_{\lambda\mu} d_1^{-1} d_2^{-1} d_\pi^{-1} + \sum_j (1 - v^X(\lambda) d_\lambda^{-1} d_1 d_2 d_\pi) \bar{\delta}_{\lambda j} f^{(1)^\dagger}(j) d_j^{-1} d_1 d_2 T^D_{N\mu}$$
$$- \sum_{\alpha j} \bar{\delta}_{\lambda\alpha} v^X(\alpha) d_\alpha^{-1} d_{\bar{j}} d_\pi \bar{\delta}_{\alpha j} f^{(1)^\dagger}(j) d_1 d_2 T^D_{N\mu} + \sum_\alpha \bar{\delta}_{\lambda\alpha} t^{(1)}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi T^D_{\alpha\mu}. \tag{146}$$

Note that once again, this equation is equivalent to that derived by AM [10] and AB [11], but with terms subtracted in order to eliminate the double-counting.

Finally, an equation for $T^D_{N\mu}$ is needed in order to close the set of equations for the $NN - \pi NN$ system. Beginning with Eq. (145) for $T^D_{N\mu}$, we use the covariant AGS equations and Eq. (137) for T^D_{NN} in order to obtain:

$$T^D_{N\mu} = \sum_j f^{(1)}(j) d_j^{-1} \bar{\delta}_{j\mu} (1 - d_1 d_2 d_\pi v^X(\mu) d_\mu^{-1}) + \bar{V} d_1 d_2 T^D_{N\mu}$$
$$+ \sum_{j\alpha} f^{(1)}(j) \bar{\delta}_{j\alpha} d_{\bar{j}} d_\pi \tilde{t}^{(1)\dagger}(\alpha) d_\alpha^{-1} d_1 d_2 d_\pi T^D_{\alpha\mu}, \tag{147}$$

with $\bar{V}$ given by Eq. (119). Note the changes in this equation as compared with AM's [10] and AB's [11] result:

1. The $t^{(1)}(\alpha)$ appearing before $f^{(1)}(j)$ has been replaced by a $\tilde{t}^{(1)\dagger}(\alpha)$.

2. V_{OPE} has been replaced by $\bar{V}$.

3. The term $\sum_j f^{(1)}(j) \bar{\delta}_{j\mu} d_{\bar{j}} d_\pi v^X(\mu) d_\mu^{-1}$ has been subtracted from the driving term.

VIII. ANTI-SYMMETRIZATION OF THE COUPLED EQUATIONS

At this stage the amplitudes T^D_{NN}, $T^D_{\lambda N}$, $T^D_{N\mu}$ and $T^D_{\lambda\mu}$ are unphysical, since they have not been anti-symmetrized. In this section we derive equations for the anti-symmetrized amplitudes. These anti-symmetrized equations apply to identical nucleons.

In order to anti-symmetrize the equations we must first derive the correct anti-symmetrization procedure. It is by no means self-evident that the ordinary non-relativistic procedure of taking appropriate linear combinations of unsymmetrized amplitudes will still be valid for these amplitudes which are fully relativistic and obey four-dimensional integral equations. The only way to discover the correct anti-symmetrization procedure is to examine the Feynman Rules for the amplitudes. Examination of the Feynman Rules for a two-fermion Green's function shows that if

$$G^D_{NN}(p'_1, p'_2; p_1, p_2) \tag{148}$$

is the Green's function which is obtained when two nucleons are treated as distinguishable particles then:

$$G_{NN}(p'_1, p'_2; p_1, p_2) = G^D_{NN}(p'_1, p'_2; p_1, p_2) - G^D_{NN}(p'_2, p'_1; p_1, p_2). \tag{149}$$

Applying LSZ reduction to each side of this equation, in order to obtain an equation connecting the amplitudes involved, then gives:

$$T_{NN} = (1 - P_{12}) T^D_{NN}, \tag{150}$$

where T^D_{NN} is the amplitude for distinguishable particles and P_{12} acting on any state interchanges the roles of particles one and two in that state. Using the fact that:

$$(1 - P_{12})^2 = 2(1 - P_{12}), \tag{151}$$

and that:

$$[P_{12}, T^D_{NN}] = 0, \tag{152}$$

Eq. (150) gives:

$$T_{NN} = \frac{1}{2}(1 - P_{12}) T^D_{NN} (1 - P_{12}). \tag{153}$$

A similar argument for the $NN \to NN\pi$ and $NN\pi \to NN\pi$ Green's functions shows:

$$F^{(1)^\dagger} = \frac{1}{2}(1 - P_{12}) F^{(1)^\dagger}_D (1 - P_{12}) \tag{154}$$

$$M^{(1)} = \frac{1}{2}(1 - P_{12}) M^{(1)}_D (1 - P_{12}). \tag{155}$$

Consequently, if we now write:

$$X^{\text{physical}}_{\Delta N} \equiv \langle \phi(2) | \langle \chi_2 | d_1 d_\pi \mathrm{Res}_{(N_1\pi)\ \text{pole}} d_1 d_\pi F^{(1)^\dagger} | \psi_D \rangle, \tag{156}$$

we find that:

$$X^{\text{physical}}_{\Delta N} = \langle \phi(2) | \langle \chi_2 | d_1 d_\pi T_{\Delta N} | \psi \rangle, \tag{157}$$

where:

$$T_{\Delta N} = \frac{1}{\sqrt{2}}(T^D_{2N} - P_{12} T^D_{1N}), \tag{158}$$

and $|\psi\rangle$ is a fully anti-symmetrized NN wave-function:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(1 - P_{12}) |\psi_D\rangle. \tag{159}$$

(Note that the choice of final-state spectator in Eq. (156) makes no difference to the result, beyond an overall minus sign.) Similarly:

$$X_{\Delta d}^{\text{physical}} \equiv \langle\phi(2)|\langle\chi_2|d_1 d_\pi \text{Res}_{(N_1\pi)\text{ pole}} d_1 d_\pi M^{(1)} d_1 d_2 \text{Res}_{(N_1 N_2)\text{ pole}} d_1 d_2 |\phi(3)_D\rangle|\chi_\pi\rangle \quad (160)$$
$$= \langle\phi(2)|\langle\chi_2|d_1 d_\pi T_{\Delta d} d_1 d_2|\phi(d)\rangle|\chi_\pi\rangle, \quad (161)$$

where:

$$T_{\Delta d} = \frac{1}{\sqrt{2}}(T_{23}^D - P_{12}T_{13}^D), \quad (162)$$
$$|\phi(d)\rangle = \frac{1}{\sqrt{2}}(1 - P_{12})|\phi(3)_D\rangle. \quad (163)$$

We now take the definitions of the distinguishable particle matrix elements X_{22}, X_{N2}, X_{32}, X_{NN}, X_{33}, X_{3N} and X_{N3}, replacing everywhere the distinguishable particle amplitudes with their identical-particle counterparts (153)–(155). This yields definitions for the identical-particle matrix elements $X_{\Delta\Delta}^{\text{physical}}$, $X_{N\Delta}^{\text{physical}}$, $X_{d\Delta}^{\text{physical}}$, X_{NN}^{physical}, X_{dd}^{physical}, X_{dN}^{physical} and X_{Nd}^{physical}. When taken together with the definitions $X_{\Delta N}^{\text{physical}}$ and $X_{\Delta d}^{\text{physical}}$ given above this produces nine definitions which may be written in the form:

$$X_{AB}^{\text{physical}} = \langle\psi_A|T_{AB}|\psi_B\rangle, \quad (164)$$

where $A, B = N, \Delta, d$,

$$|\psi_N\rangle = |\psi_{NN}\rangle \quad (165)$$
$$|\psi_\Delta\rangle = d_1 d_\pi|\phi(2)\rangle|\chi_2\rangle \quad (166)$$
$$|\psi_d\rangle = d_1 d_2|\phi(d)\rangle|\chi_\pi\rangle; \quad (167)$$

and:

$$T_{NN} = T_{NN}^D; \quad T_{Nd} = T_{N3}^D; \quad T_{dN} = T_{3N}^D; \quad T_{dd} = T_{33}^D; \quad (168)$$
$$T_{\Delta N} = \frac{1}{\sqrt{2}}(T_{2N}^D - P_{12}T_{1N}^D); \quad T_{N\Delta} = \frac{1}{\sqrt{2}}(T_{N2}^D - T_{N1}^D P_{12}); \quad (169)$$
$$T_{\Delta d} = \frac{1}{\sqrt{2}}(T_{23}^D - P_{12}T_{13}^D); \quad T_{d\Delta} = \frac{1}{\sqrt{2}}(T_{32}^D - T_{31}^D P_{12}); \quad (170)$$
$$T_{\Delta\Delta} = T_{22}^D - T_{21}^D P_{12}. \quad (171)$$

Equations for these amplitudes may then be derived from the Eqs. (134), (137), (145) and (146) for $T_{\lambda N}^D$, T_{NN}^D, $T_{N\mu}^D$ and $T_{\lambda\mu}^D$. We find that the set of coupled equations:

$$T_{\Delta N} = \sqrt{2}\left[f^\dagger(1)d_1^{-1} - Y\right](1 + d_1 d_2 T_{NN}) - P_{12}t(2)d_1 d_\pi T_{\Delta N} + \sqrt{2}t(3)d_1 d_2 T_{dN}, \quad (172)$$
$$T_{dN} = \left[\sum_k f^\dagger(k)d_k^{-1} - Y\right](1 + d_1 d_2 T_{NN}) + \sqrt{2}t(2)d_1 d_\pi T_{\Delta N}, \quad (173)$$
$$T_{NN} = \bar{V}(1 + d_1 d_2 T_{NN}) + \sqrt{2}f(1)d_2 d_\pi \tilde{t}^\dagger(2)d_1 d_\pi T_{\Delta N} + \sum_k f(k)d_{\bar{k}} d_\pi t(3)d_1 d_2 T_{dN}, \quad (174)$$

describe the coupled $NN \to \pi NN$ and $NN \to NN$ two-fragment processes. Meanwhile the equations:

$$
\begin{aligned}
T_{\Delta\Delta} &= -P_{12} d_1^{-1} d_2^{-1} d_\pi^{-1} + \sqrt{2}\left[f^\dagger(1) d_1^{-1} - Y\right] d_1 d_2 T_{N\Delta} \\
&\quad - P_{12} t(2) d_1 d_\pi T_{\Delta\Delta} + \sqrt{2} t(3) d_1 d_2 T_{d\Delta} && (175) \\
T_{d\Delta} &= \sqrt{2} d_1^{-1} d_2^{-1} d_\pi^{-1} + \left[\sum_k f^\dagger(k) d_k^{-1} - Y\right] d_1 d_2 T_{N\Delta} + \sqrt{2} t(2) d_1 d_\pi T_{\Delta\Delta} && (176) \\
T_{N\Delta} &= \sqrt{2} f(1) d_1^{-1}(1 - d_1 d_\pi v^X(2)) + \bar{V} d_1 d_2 T_{N\Delta} + \sqrt{2} f(1) d_2 d_\pi \tilde{t}^\dagger(2) d_1 d_\pi T_{\Delta\Delta} \\
&\quad + \sum_k f(k) d_{\bar{k}} d_\pi t(3) d_1 d_2 T_{d\Delta} && (177) \\
T_{\Delta d} &= \sqrt{2} d_1^{-1} d_2^{-1} d_\pi^{-1} + \sqrt{2}\left[f^\dagger(1) d_1^{-1} - Y\right] d_1 d_2 T_{Nd} - P_{12} t(2) d_1 d_\pi T_{\Delta d} \\
&\quad + \sqrt{2} t(3) d_1 d_2 T_{dd} && (178) \\
T_{dd} &= \left[\sum_k f^\dagger(k) d_k^{-1} - Y\right] d_1 d_2 T_{Nd} + \sqrt{2} t(2) d_1 d_\pi T_{\Delta d} && (179) \\
T_{Nd} &= \sum_k f(k) d_k^{-1} + \bar{V} d_1 d_2 T_{Nd} + \sqrt{2} f(1) d_2 d_\pi \tilde{t}^\dagger(2) d_1 d_\pi T_{\Delta d} \\
&\quad + \sum_k f(k) d_{\bar{k}} d_\pi t(3) d_1 d_2 T_{dd}, && (180)
\end{aligned}
$$

describe the $NN\pi \to NN\pi$ and $NN \to NN\pi$ two-fragment processes. Here:

$$
Y = \sum_k v^X(\bar{k}) d_\pi f^\dagger(k). \tag{181}
$$

Note that the t-matrices and vertices used as input to these equations are always the one-particle irreducible amplitudes, $t^{(1)}$, $f^{(1)}$ and $f^{(1)\dagger}$. An apparent exception is the inclusion of the modified one-particle irreducible t-matrix $\tilde{t}^{(1)\dagger}$, which is used in some places in the equations. But, $\tilde{t}^{(1)\dagger}$ is, of course, defined in terms of $t^{(1)}$ and $f^{(1)}$ by Eqs. (64) and (59).

IX. SPECIFICATION OF INPUT AMPLITUDES AND IMPLICIT INCLUSION OF OTHER HADRONIC STATES

Equations (172)–(180) are a set of two-fragment scattering equations for the processes:

$$
\left.\begin{array}{r} N + N + \pi \\ N + N \end{array}\right\} \longrightarrow \left\{\begin{array}{l} N + N + \pi \\ N + N \end{array}\right. \tag{182}
$$

Each Feynamn diagram in which no more than one explicit pion appears is included once and only once in the equations. Hence these four-dimensional $NN - \pi NN$ equations obey two and three-body unitarity and contain no double-counting, thus providing a framework in which predictions for scattering observables in the $NN - \pi NN$ system can be made.

The first step in making such predictions is the specification of the input amplitudes $t_{\pi N}^{(1)}$ and $f^{(1)}$. In Appendix B the one-explicit-pion content of $t_{\pi N}^{(1)}$ and $f^{(1)}$ is elucidated by the Taylor method, demonstrating that the ampitudes which need to be specified are, in fact,

$t^{(2)}_{\pi N}$ and $f^{(2)}$. Now, $t^{(2)}_{\pi N}$ and $f^{(2)}$ could have their *two*-explicit-pion content analyzed (see e.g., [33] for the πN case in time-ordered perturbation theory). However, even were we able to solve the resulting set of coupled $\pi N - \pi\pi N$ equations together with Equations (172)–(180) the results would still only include the effects of pion exchange. It is well known that other effects, e.g. heavy-meson exchange and delta degrees of freedom, are important in nuclear forces, and it is not clear how such mechanisms would be included in this description.

Therefore in order to implicitly include some of this additional physics we advocate calculating $f^{(2)}$ and $t^{(2)}_{\pi N}$ in some other (hadronic or quark) model and/or parametrizing them in such a way as to simplify the solution of Eqs. (172)–(180). Note that since, by definition, these amplitudes have no one-explicit-pion content, such an approach does not affect NN or πNN unitarity. Nor, if it is implemented correctly, should it introduce any double-counting.

Firstly, the bare πNN form factor $f^{(2)}$ may be taken from some model of QCD, as long as care is taken with any pionic corrections so that overcounting is avoided. Alternatively, $f^{(2)}$ may simply be parametrized.

Secondly, two approaches may be taken with $t^{(2)}_{\pi N}$. On the one hand, a separable potential may be postulated for $t^{(2)}_{\pi N}$ and the free parameters of the separable form fitted to the πN scattering data. (Note that it may be necessary to use πN scattering data in the P_{11} channel in order to fix any parameters in $f^{(2)}$.) In this way known mechanisms of the $\pi - N$ interaction, such as ρ and ω exchange and the Δ resonance, are implicitly included insofar as they contribute to the actual data. On the other hand, if we wish to know *exactly* what physics is included in the input πN interaction a meson exchange model of πN scattering, with the Δ resonance built in, may be constructed (see e.g. [34]) and a separable expansion of the resulting amplitude made, via the Ernst-Shakin-Thaler technique [35,36]. Such an expansion has already been accomplished for the three-dimensional Paris NN potential [37]. In the πN case Pearce and Afnan have used the extension of Ernst, Shakin and Thaler's technique developed by Pearce [38] in order to obtain a separable expansion of the πN interaction in the Cloudy Bag Model [39]. The work of Rupp and Tjon [40] shows that making a separable expansion of a Bethe-Salpeter amplitude is not computationally intensive, thus once a meson-exchange model of the πN system has been used to calculate scattering via the Bethe-Salpeter Equation a separable expansion of the resulting amplitude should not be a difficult task.

Thirdly, so far we have regarded $T^{(1)}_{NN}$ as part of the solution to Eqs. (172)–(180). However, this amplitude is also needed in order to construct the kernel of these equations. Indeed, it is $T^{(1)}_{NN}$s appearance in the kernel (as $t^{(1)}_{NN}$) which ensures that higher pionic processes, such as crossed two-pion exchange, are included in the solution to the coupled equations (see Figure 18). In theory it is possible to "bootstrap" this theory up and so generate an exact solution to the non-linear equation for $T^{(1)}_{NN}$. However even if this could be achieved, important physics would be missing from the resulting NN interaction. In order to simultaneously include some of these missing mechanisms and snap the "bootstrap", we advocate the same approach to the input $t^{(1)}_{NN}$ as to $t^{(1)}_{\pi N}$. Either Eq. (11) for $T^{(1)}_{NN}$ should be used and a separable potential for $T^{(2)}_{NN}$ fitted to the experimental data, or a four-dimensional covariant separable expansion of some meson-exchange NN amplitude should be made. As in the πN case, since the input $t^{(1)}_{NN}$ occurs only in the presence of a spectator pion such an approach will not destroy NN or $NN\pi$ unitarity, nor will it, if carefully implemented, introduce double-

counting. The consistency of such a procedure can be checked by comparing the solution $T_{NN}^{(1)}$ with the input $t_{NN}^{(1)}$.

If the amplitudes $t_{NN}^{(1)}$, $t_{\pi N}^{(1)}$ and $f^{(1)}$ are constructed in this way then (provided a separable expansion for v^X is also made) Eqs. (172)–(180) become a set of coupled Bethe-Salpeter equations for the $NN - \pi NN$ system. This is completely analagous to the situation in the non-relativistic three-body Faddeev equations [41]. By using separable expansions for the input amplitudes these may be reduced to the Lovelace equations [42], which are a set of coupled two-body equations.

One question which must be answered is which energy domain any separable expansion of $t_{NN}^{(1)}$ and $t_{\pi N}^{(1)}$ should be made in. Suppose that Figure 19 is a piece of a Feynman Diagram summed in the four-dimensional $NN - \pi NN$ equations, with α the spectator particle label ($\alpha = \pi, N$) and k_α its four-momentum. The analytic expression for this Feynman diagram would therefore contain a piece:

$$\int d^4 k_\alpha \cdots G_{\bar{\alpha}}(P - k_\alpha)\, t_\alpha^{(1)}(s_\alpha)\, d_\alpha(k_\alpha)\, G_{\bar{\alpha}}(P - k_\alpha) \cdots \tag{183}$$

where

$$P = (\sqrt{s}, \vec{0}) \tag{184}$$

$$s_\alpha = (P - k_\alpha)^2 = (\sqrt{s} - k_\alpha^0)^2 - k_\alpha^2 \tag{185}$$

is the total four-momentum in the three-body centre of mass, and $G_{\bar{\alpha}}(P - k_\alpha)$ is the product of the two propagators of the interacting particles.

Since $k_\alpha^0 \in (-\infty, \infty)$ the two-body energy which is the argument of $t_\alpha^{(1)}$ ranges from $-\infty$ to ∞. However, suppose that a separable expansion for $t_\alpha^{(1)}$ has been made, i.e.:

$$t_\alpha^{(1)}(s_\alpha) = \sum_i g_i^\dagger \frac{1}{s_\alpha^+ - m_i^2} g_i, \tag{186}$$

with g_i an appropriate form factor implicitly dependent on the relative four-momentum of the two particles in the interacting subsystem. In order to perform the k_α^0 integration we examine the analytic structure of the integrand in the complex k_α^0 plane.

Apart from the dependence listed in Eq. (183), the only place k_α appears in the integrand is via its presence in the form factors which occur immediately preceding and following the piece of diagram which is explicitly written in (183). Unless these are πNN form factors their analytic structure in the relative momentum variable must give no contribution to unitarity, and therefore their k_α-dependence may be ignored. If one (or both) form factors is a πNN form factor then there is a πN cut (or cuts) to be added to the analytic structure in the lower half of the k_α^0 plane that is listed below. However, the existence of such cuts makes no difference to the final conclusion obtained here. Consequently, for the purposes of this argument, we ignore any k_α-dependence of the form factors.

The poles of the integrand in the lower half-plane are at:

$$\sqrt{m_\alpha^2 + |\vec{k}_\alpha|^2} - i\epsilon \tag{187}$$

$$\sqrt{s} + \sqrt{m_i^2 + |\vec{k}_\alpha|^2} - i\epsilon \tag{188}$$

while the two-body propagator contributes a cut beginning at:

$$\sqrt{s} + m_{\bar{\alpha}} - i\epsilon \tag{189}$$

where $m_{\bar{\alpha}}$ is the mass of the two-body system. Provided that s is in the region for particle-particle scattering the piece of analytic structure giving the most significant contribution to the integral is the pole (187).

Thus the important two-body energies s_α are those satisfying $\sqrt{s_\alpha} < \sqrt{s} - m_\alpha$, in agreement with the case of non-relativistic few-body physics [17]. Therefore if $\sqrt{s}$ is restricted to:

$$2m \leq \sqrt{s} \leq 2m + 2\mu \tag{190}$$

then πN and NN amplitude parametrizations based on information in the region below the respective pion production thresholds should give accurate results when used in the solution of Eqs. (172)–(180).

Finally, it is clear that NN interactions mediated by meson exchanges which are totally non-pionic are not included in our equations. These exchanges are known to be important effects in the medium to short range part of the nucleon-nucleon interaction. They could be included by extending the arguments given here for the pion in order to expose one-explicit-meson states for *all* mesons of importance. However, at this stage such an approach would be unnecessarily thorough, since we do not wish to make predictions for the reactions:

$$NN + X \leftrightarrow NN \tag{191}$$

$$X + d \rightarrow X + d, \tag{192}$$

where X is any meson known to be important in one-boson-exchange NN potentials! Therefore we can afford to merely add a phenomenological heavy-meson exchange piece to the $T_{NN}^{(2)}$ given by Eq.(117), as was done in the three-dimensional $NN - \pi NN$ case by Avishai and Mizutani [10,23]. Note that some care must be taken in making this addition. For instance, the piece of ρ exchange arising from the exchange of two uncorrelated pions is separately included in the $NN - \pi NN$ equations, therefore if a ρ is phenomenologically added to our formalism it will not necessarily have the same parameters as in NN OBE potentials.

X. CONCLUSION

In this paper we have derived coupled four-dimensional covariant scattering equations for the $NN - \pi NN$ system, using the modified Taylor method of classification of diagrams [21]. These equations sum all the covariant perturbation theory diagrams which include one explicit pion once and only once. They therefore:

1. Are four-dimensional integral equations.

2. Are covariant, not only on-shell, but also have off-shell covariance in the manner dictated by the Feynman graphical expansion.

3. Are completely free of the double-counting problems of some previous four-dimensional $NN - \pi NN$ equations.

4. Obey NN and $NN\pi$ unitarity, by explicit construction.

The double-counting subtractions found to be necessary are a consequence of attempting to derive a set of coupled equations which describe, in a unified way, the reactions:

$$\begin{aligned} N + N &\rightarrow N + N \\ N + N &\leftrightarrow N + N + \pi \\ N + N + \pi &\rightarrow N + N + \pi, \end{aligned} \tag{193}$$

by summing all covariant perturbation theory Feynman diagrams for the processes (193) that involve one explicit pion or no explicit pions. Were one content merely to take the Bethe-Salpeter equation for NN scattering and choose a set of Feynman diagrams for the kernel $T_{NN}^{(2)}$ no double-counting would arise. However, such an approach does not satisfy three-body unitarity, since it excludes certain s-channel two-particle irreducible diagrams which contain one explicit pion. Only by deriving a set of coupled equations which sum *all* covariant perturbation theory diagrams containing zero or one explicit pion(s), as we have done here, will three-body unitarity be obeyed.

The modified Taylor method used to make this derivation is completely general and so could also be used to obtain equations for other few-hadron processes. For example, the $\pi N - \pi\pi N$ problem and pion photoproduction would both be amenable to a derivation by this method.

The double-counting exposed and eliminated here occurred in four-dimensional $NN - \pi NN$ equations. The implication of this double-counting for the "standard" three-dimensional $NN - \pi NN$ equations is not entirely clear. On the one hand, an obvious way to obtain three-dimensional equations from the ones given here is to take the two and three-particle Green's functions appearing in these equations and replace them by three-dimensional Green's functions, as suggested by, for example, Blankenbecler and Sugar [22]. It is apparent that the equations thus produced will be different to the standard $NN - \pi NN$ equations, since they will still include subtractions for double-counting. On the other hand, the standard $NN - \pi NN$ equations may also be derived by classifying diagrams in time-ordered perturbation theory, as was mentioned in the Introduction. If this approach is taken the resulting equations do not double-count any time-ordered perturbation theory diagrams. Hence, one may question whether the subtractions which would appear were one to apply a three-dimensional reduction to our $NN - \pi NN$ equations are really necessary.

Subject to this issue of the relation to a three-dimensional calculation, our results may have ramifications beyond few-hadron processes. For instance, the double-counting present in the unsubtracted four-dimensional $NN - \pi NN$ equations may have implications for pion absorption and scattering on larger nuclei, as has been discussed by, for example Kowalski et al. [30]. Therefore the remedies for this double-counting discussed above may well prove applicable in these larger systems.

The set of equations obtained here are (apart from the minor point discussed in Section VI B 1) equivalent to those previously obtained by Kvinikhidze and Blankleider (KB) [24,25], although we have anchored our derivation more firmly in Taylor's original work. Furthermore, the final set of coupled equations given here are ready for computation, being written in the form of two-fragment scattering equations.

These scattering equations require as their input only the $\pi - N$ t-matrix and dressed πNN vertex. However, since the NN t-matrix is present in the equations both within the

kernel and as part of the solution it appears that some way of "bootstrapping" the theory up must be found. In Section IX we discussed how this bootstrap might be avoided and came to the inclusion that making a separable expansion of a model NN amplitude at energies up to the pion-production threshold should provide a good input NN amplitude. Even once the bootstrap problem is resolved in this way the computation is still somewhat formidable, involving, as it does, the numerical solution of a set of coupled four-dimensional integral equations. Nevertheless, the equations derived here (together with those obtained by KB) represent the first complete and correct summation of the one-explicit-pion sector of the theory of the $NN - \pi NN$ system. Not until these equations are solved will it be clear how much of the physics of the $NN - \pi NN$ system is attributable to the one-explicit-pion sector. Therefore, we believe that the numerical work involved here, although at first sight somewhat daunting, is necessary.

Despite the fact that we have had to expose some $NN\pi\pi$ states to remove double-counting on the $NN\pi$ level we have, in general, refrained from entering the two-pion sector of the theory. This has meant that the amplitudes $M^{(3)}$, ${F^{(3)}}^\dagger$ and $T^{(3)}$ have been completely ignored. The modification to the above equations to include such "three-body" mechanisms is one obvious way in which these equations could be extended.

We note that mesons other than the pion and baryonic states such as the Δ may be implicitly included in the equations via their (possibly parametrized) presence in input amplitudes. They may be included explicitly if the Lagrangian is extended to one involving all hadrons of interest. Such a modification of the underlying field theory merely results in a proliferation of possible diagrams, rather than any fundamental change in the way the derivation proceeds.

However, such improvements are in the future. Until the equations as they stand here are solved, and it is seen how well they reproduce the experimental data in the $NN - \pi NN$ system, we shall not know which, if any, of these more sophisticated mechanisms are necessary for a correct description of the $NN - \pi NN$ system dynamics. The next stage of this work must therefore be an effort to solve the equations given here numerically.

As a first step towards this goal we have recently used the formalism developed here to perform a calculation of $\phi\phi$ scattering in a scalar $\phi^2\sigma$ field theory. Numerical methods and results will be detailed in a later paper [43].

APPENDIX A: PROPAGATOR DRESSING

In this appendix we discuss how to accomplish the dressing of all the particles involved. The procedure used is similar to that developed by Afnan and Blankleider [11,17].

Consider the amplitude $A_{n\leftarrow m}$ for the $m \rightarrow n$ transition. The diagrams contributing to A may be of two types:

1. Those for which a self-energy contribution on one leg, which we consider to be leg i in the initial state, may be isolated. (The argument is exactly the same for self-energy contributions on legs in the final state.) These diagrams are of the general form:

$$A_{n\leftarrow m} d_i^{(0)} \Sigma_i, \tag{A1}$$

where $d_i^{(0)}$ and Σ_i are the undressed propagator and self-energy of particle i. (If there are j nucleons in the initial state then for $i \leq j$ $d_i^{(0)}$ and Σ_i represent a nucleon propagator and self-energy, otherwise they represent a pion propagator and self-energy.) See Figure 20 for an example of such a diagram.

2. The diagrams which cannot have a self-energy contribution isolated in this way. We denote these by:

$$A_{n\leftarrow m(\tilde{i})} \tag{A2}$$

Therefore,

$$A_{n\leftarrow m} = A_{n\leftarrow m} d_i^{(0)} \Sigma_i + A_{n\leftarrow m(\tilde{i})}, \tag{A3}$$

$$\Rightarrow A_{n\leftarrow m} d_i^{(0)} = A_{n\leftarrow m(\tilde{i})} d_i. \tag{A4}$$

Since,

$$d_i = d_i^{(0)}(1 - \Sigma_i d_i^{(0)})^{-1} = (d_i^{(0)^{-1}} - \Sigma_i)^{-1} = Z_i d_i^R, \tag{A5}$$

where Z_i is the wave function renormalization for particle i, and d_i^R is the renormalized propagator with unit residue at the pole of d_i which corresponds to the physical mass of particle i.

Repeating this procedure for all external legs leads to the following result:

$$G_n A_{\tilde{n}\leftarrow\tilde{m}} G_m = G_n^{(0)} A_{n\leftarrow m} G_m^{(0)}, \tag{A6}$$

where:

$$G_k = \prod_{i=1}^{k} d_i Z_i^{-\frac{1}{2}}, \tag{A7}$$

$$G_k^{(0)} = \prod_{i=1}^{k} d_i^{(0)}, \tag{A8}$$

and $A_{\tilde{n}\leftarrow\tilde{m}}$ has had all the external bubbles removed from it, and factors of $Z^{\frac{1}{2}}$ included in it, in order to make it agree with Eq. (7).

This result allows us to consider all initial and final state legs as fully-dressed, provided that we work with the amplitude $A_{\tilde{n}\leftarrow\tilde{m}}$ which has no bubbles on these initial or final state legs. Bubbles may also be eliminated from the internal legs of $A_{\tilde{n}\leftarrow\tilde{m}}$ by a similar procedure, as follows. Consider any set of diagrams contributing to $A_{\tilde{n}\leftarrow\tilde{m}}$ which are $(k-1)$-particle irreducible but admit an internal k-particle cut. According to the last internal cut lemma these diagrams may be expressed uniquely as:

$$A_{\tilde{n}\leftarrow k} G_k^{(0)} A_{\tilde{k}\leftarrow\tilde{m}}. \tag{A9}$$

Using the same arguments as above any bubbles appearing in the initial state of $A_{\tilde{n}\leftarrow k}$ may be amputated and placed in G_k. Consequently, we obtain for this sum:

$$A_{\tilde{n}\leftarrow\tilde{k}} G_k A_{\tilde{k}\leftarrow\tilde{m}}. \tag{A10}$$

Therefore we may always work with fully-dressed particles, provided that we consider all amplitudes to be of the type $A_{\tilde{n}\leftarrow\tilde{m}}$. In this paper we have adopted this approach, and so all amplitudes are of the $A_{\tilde{n}\leftarrow\tilde{m}}$ type, even though the tildes are never displayed explicitly.

APPENDIX B: INPUT TO THE $NN - \pi NN$ EQUATIONS: THE πNN VERTEX AND THE $\pi - N$ AMPLITUDE.

In this appendix we consider the two amplitudes which are required as input to the $NN - \pi NN$ equations: the one-particle irreducible πNN vertex, $f^{(1)}$, and the one-particle irreducible $\pi - N$ t-matrix, $t^{(1)}_{\pi N}$.

Firstly, consider $f^{(1)}$. Our aim is to apply the Taylor method to this vertex in order to derive an integral equation for it. Since we are examining the two-cut structure of $f^{(1)}$, we have $m = 2$, $n = 1$ and $r = 2$. Consequently, since we are dealing with fully-dressed particles all one-to-one amplitudes are zero and so classes C_3, C_4 and C_5 are all empty. Therefore double-counting in classes C_4 and C_5 cannot arise, even though $n \leq r$. The sum of class C_1 in this case is clearly the two-particle irreducible πNN vertex $f^{(2)}$. The sum of class C_2 is found, via the last internal cut lemma, to be:

$$f^{(2)} d_N d_\pi t^{(1)}_{\pi N}. \tag{B1}$$

Therefore, Taylor's method gives the following equation for $f^{(1)}$:

$$f^{(1)} = f^{(2)} + f^{(2)} d_N d_\pi t^{(1)}_{\pi N}. \tag{B2}$$

Taylor's method could now be used in order to extract the structure of $f^{(2)}$, but we prefer to not consider three-body unitarity for this amplitude. Instead the expression for $f^{(2)}$ will be extracted from the Lagrangian under consideration.

However, we do wish to discuss the two-particle cut structure of the 1PI $\pi - N$ amplitude, $t^{(1)}_{\pi N}$, using the Taylor method. Observe that classes C_3, C_4 and C_5 are again empty, since all particles are fully dressed. Applying the Taylor method, it is clear that the sum of class C_1 is the 2PI amplitude $t^{(2)}_{\pi N}$. Also, the LICL may be applied to class C_2 in order to obtain:

$$t^{(2)}_{\pi N} d_N d_\pi t^{(1)}_{\pi N}; \tag{B3}$$

and therefore the following Bethe-Salpeter type equation for $t^{(1)}_{\pi N}$ is obtained:

$$t^{(1)}_{\pi N} = t^{(2)}_{\pi N} + t^{(2)}_{\pi N} d_N d_\pi t^{(1)}_{\pi N}. \tag{B4}$$

As mentioned above in the case of $f^{(2)}$, the structure of $t^{(2)}_{\pi N}$ may be investigated using similar techniques as have been used for $t^{(1)}_{\pi N}$. On the other hand, if we are not concerned with $\pi\pi N$ unitarity in the $\pi - N$ amplitude we may just extract some model-dependent result for $t^{(2)}_{\pi N}$ from the Lagrangian under consideration.

The amplitude $t^{(1)}_{\pi N}$ is, however, only part of the full $\pi - N$ amplitude, $t^{(0)}_{\pi N}$. Applying Taylor's method to $t^{(0)}_{\pi N}$ in the same fashion as above, reveals that:

$$t^{(0)}_{\pi N} = t^{(1)}_{\pi N} + {f^{(1)}}^\dagger d_N f^{(0)}. \tag{B5}$$

A brief examination of the zero-particle irreducible πNN amplitude, $f^{(0)}$, shows that the only Taylor class of diagrams contributing to $f^{(0)}$ which is non-empty is C_1, which sums to give the 1PI πNN amplitude, $f^{(1)}$. Therefore,

$$f^{(0)} = f^{(1)}. \tag{B6}$$

Consequently, the full $\pi - N$ amplitude $t^{(0)}_{\pi N}$ is given by:

$$t^{(0)}_{\pi N} = t^{(1)}_{\pi N} + {f^{(1)}}^{\dagger} d_N f^{(1)}, \tag{B7}$$

where $t^{(1)}_{\pi N}$ is the non-pole part of the $\pi - N$ t-matrix, given by Eq. (B4), and ${f^{(1)}}^{\dagger} d_N f^{(1)}$ is the pole part of the $\pi - N$ t-matrix.

APPENDIX C: DETAILS OF THE CALCULATION OF $c_1 \cap C_5^{\{N2'\}}$

The portion of $\tilde{t}^{(2)}_{\pi N}$ which is 2PR in the $N' \leftarrow \pi' + \pi + N$ channel may be written, using the reverse of the LICL, the "first internal cut lemma", as:

$$f^{(1)}_{\pi_1} d_1 d_{\pi_1} P^{\dagger} \tag{C1}$$

where $P^{\dagger}$ is an $N\pi_2 \rightarrow N\pi_1\pi_2$ amplitude which is two-particle irreducible in the s-channel and the channel:

$$\pi_1' + N' \leftarrow N + \pi_2 + \pi_2',$$

as well as being 1PI in the channel:

$$\pi_2' + N \leftarrow N' + \pi_2 + \pi_1'.$$

Consequently, we might expect the diagrams:

$$f^{(1)}_{\pi_2}(1) d_{\pi_2} f^{(1)}_{\pi_1}(2) d_1 d_{\pi_1} P^{\dagger}(2) d_2 d_{\pi_2} {f^{(1)}_{\pi_2}}^{\dagger}(1) \tag{C2}$$

and:

$$f^{(1)}_{\pi_2}(1) d_{\pi_2} f^{(1)}_{\pi_1}(2) d_1 d_{\pi_1} P^{\dagger}(2) d_1 d_{\pi_2} \tilde{t}^{(1)}_{\pi_2 N}(2) d_2 d_{\pi_2} {f^{(1)}_{\pi_2}}^{\dagger}(1) \tag{C3}$$

to be included in $c_1 \cap C_5^{\{N2'\}}$. Consider first (C2). If this term is to be part of $C_5^{\{N2'\}}$ the pion absorbed on $N2'$ must be "hidden" in another amplitude, since if the diagram is to contribute to $C_5^{\{N2'\}}$ it must take the form:

$$f^{(1)}(2) d_1 d_{\pi} {F^{(2)}}^{\dagger}. \tag{C4}$$

There are three ways of "hiding" the pion:

1. Place it in an $N - N$ t-matrix. Taking this approach shows that the diagram:

$$f^{(1)}_{\pi}(1) d_{\pi} f^{(1)}_{\pi'}(2) d_1 t^{(1)}_{\pi N}(2) d_{\pi'} d_1 {f^{(1)}_{\pi'}}^{\dagger}(2) d_{\pi} d_2 {f^{(1)}_{\pi}}^{\dagger}(1) \tag{C5}$$

is in $c_1 \cap C_5^{\{N2'\}}$, since it occurs in both c_1 and the term:

$$f^{(1)}(2) d_{\pi} d_1 T^{(1)}_{NN} d_1 {f^{(1)}}^{\dagger}(2) \tag{C6}$$

of $C_5^{\{N2'\}}$.

2. The second way of hiding the second pion is to place it in the $\pi - N$ t-matrix. That is, to note that the diagrams:

$$f_{\pi_2}^{(1)}(1)d_{\pi_2}d_2f_{\pi_1}^{(1)}(2)d_{\pi_1}d_1t_{\pi_1\pi_2}^{(1)}d_{\pi_1}f_{\pi_1}^{(1)^\dagger}(2)d_{\pi_2}f_{\pi_2}^{(1)^\dagger}(1) \text{ and}$$
$$f_{\pi_2}^{(1)}(1)d_{\pi_2}d_2f_{\pi_1}^{(1)}(2)d_{\pi_1}d_1t_{\pi_1\pi_2}^{(1)}d_{\pi_2}f_{\pi_1}^{(1)^\dagger}(2)d_{\pi_1}f_{\pi_2}^{(1)^\dagger}(1) \tag{C7}$$

is included in both c_1 and:

$$f^{(1)}(2)d_\pi d_2 t_{\pi N}^{(1)}(1)d_\pi f^{(1)^\dagger}(2), \tag{C8}$$

which is part of $C_5^{\{N2'\}}$.

3. Finally, one can "hide" the pion in a three-body force. This suggests that if the term:

$$f^{(1)}(2)d_\pi d_1 M_1^{(3)} d_2 d_\pi f^{(1)^\dagger}(1) \tag{C9}$$

is included in $C_5^{\{N2'\}}$ then a number of diagrams in c_1 will also occur in $C_5^{\{N2'\}}$. Precisely which diagrams from c_1 are included in (C9) though? The additional irreducibility constraints placed on $M_1^{(3)}$ mean that only that part of c_1, $\tilde{c}_1$, which is given by:

$$\tilde{c}_1 = f_{\pi_2}^{(1)}(1)d_{\pi_2}d_2f^{(1)}(2)d_1d_{\pi_1}\tilde{P}^\dagger(2)d_{\pi_2}f_{\pi_2}^{(1)^\dagger}(1), \tag{C10}$$

where $\tilde{P}^\dagger$ is 1PI in both the channels:

$$N' + N \leftarrow \pi_1' + \pi_2' + \pi_2 \text{ and } \pi_1' + N \leftarrow N' + \pi_2' + \pi_2,$$

is included in (C9). The contributions to c_1 produced by the parts of $P^\dagger$ which are 1PR in these two channels must be dealt with separately. This, in fact, is the reason why possibilities 1 and 2 had to be dealt with above. Once this is done, however, only the part $\tilde{c}_1$ of c_1 is left for consideration. But, $\tilde{c}_1 \subset c_1 \cap C_5^{\{N2'\}}$ if and only if (C9) is included in $C_5^{\{N2'\}}$. Consequently, in this calculation, in which $M_1^{(3)}$ is set to zero, the diagrams $\tilde{c}_1$ are not in $c_1 \cap C_5^{\{N2'\}}$.

Turning now to (C3), similar arguments to the above show that the expression (C3) is only part of $C_5^{\{N2'\}}$ if a pure three-body force is included in the calculation. Therefore, for the purposes of this argument (C3) is *not* in $C_5^{\{N2'\}} \cap c_1$.

APPENDIX D: DETAILS OF THE CALCULATION OF $c_2 \cap C_5^{\{N2'\}}$

The first internal cut lemma may be used to show that the diagrams which contribute to $\tilde{T}_{NN}^{(2)}$ and are 2PR in the $N1' \leftarrow N1 + N2 + N2'$-channel sum to give:

$$f^{(1)}(2)d_1d_\pi\tilde{F}^{(2)^\dagger}, \tag{D1}$$

where $\tilde{F}^{(2)\dagger}$ is two-particle irreducible in the s-channel and the channel:

$$\pi' + N1' \leftarrow N1 + N2 + N2',$$

as well as being 1PI in the channel:

$$N2' + N2 \leftarrow N1' + \pi' + N1.$$

The question now is, which portions of the two expressions:

$$f_{\pi_1}^{(1)}(1) d_2 d_{\pi_1} f_{\pi_2}^{(1)}(2) d_2 d_{\pi_2} \tilde{F}^{(2)^\dagger} d_2 f_{\pi_1}^{(1)^\dagger}(1) \tag{D2}$$

and:

$$f_{\pi_1}^{(1)}(1) d_2 d_{\pi_1} f_{\pi_2}^{(1)}(2) d_2 d_{\pi_2} \tilde{F}^{(2)^\dagger} d_1 d_2 T_{NN}^{(1)} d_2 f_{\pi_1}^{(1)^\dagger}(1) \tag{D3}$$

are also in $C_5^{\{N2'\}}$? As in Appendix C we argue that we are not interested in those portions of expressions (D2) and (D3) which may be reexpressed in the form (C9). This line of reasoning leads to the conclusion that the diagrams contained in (D2) and $C_5^{\{N2'\}}$ are:

$$f_{\pi_1}^{(1)}(2) d_{\pi_1} f_{\pi_2}^{(1)}(1) d_2 t_{\pi_1 N}^{(1)}(1) d_2 d_{\pi_2} f_{\pi_2}^{(1)^\dagger}(1) d_1 d_{\pi_1} f_{\pi_1}^{(1)^\dagger}(2), \tag{D4}$$

and:

$$f_{\pi_1}^{(1)}(2) d_1 d_{\pi_1} f_{\pi_2}^{(1)}(1) d_{\pi_2} d_1 \tilde{T}_{NN}^{(1)} d_1 f_{\pi_2}^{(1)^\dagger}(1) d_2 d_{\pi_1} f_{\pi_1}^{(1)^\dagger}(2), \tag{D5}$$

where $\tilde{T}_{NN}^{(1)}$ is not only one-particle irreducible in the s-channel, but also 1PI in the t-channel.

Similarly, there are no diagrams which are in both (D3) and $C_5^{\{N2'\}}$ provided, once again, that no pure three-body force is included in the calculation.

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FIGURES

FIG. 1. The term on the right is included in the current $NN-\pi NN$ equations, while the term on the left, known as the Jennings term, is not included.

FIG. 2. One covariant perturbation theory diagram which is double-counted if the original Taylor method is used to construct an equation for $T_{NN}^{(2)}$, with the two cuts which place it in both $C_3^{\{N1\}}$ and $C_3^{\{N2\}}$.

FIG. 3. If the two cuts shown are three-cuts then they will place certain diagrams which should only be included in C_4 in C_5 as well.

FIG. 4. One possible source of double-counting: if $t_{\pi N}^{(2)}$ is one-particle reducible in the u-channel then the cut shown places this diagram in $C_4^{\{\pi'\}\{N2\}}$. Since the diagram arose as part of $C_4^{\{N2'\}\{N1\}}$ such u-channel one-particle reducibility leads to double-counting.

FIG. 5. Another possible source of double-counting: if $t_{\pi N}^{(2)}$ is one-particle reducible in the t-channel then the cut shown places this diagram in $C_4^{\{N2'\}\{N1\}}$. Since the diagram arose as part of sub-class $C_4^{\{N1'\}\{N2\}}$ such t-channel one-particle reducibility would lead to double-counting.

FIG. 6. An example of a diagram which would lead to double-counting of the type depicted in Figure 5.

FIG. 7. The first term in the sub-class $C_5^{\{N1'\}}$, c_1, with the cut which may lead to double-counting indicated.

FIG. 8. The lowest order diagrams which contribute to $\pi - \pi$ scattering in a Lagrangian with a πNN vertex and a πN contact term.

FIG. 9. The second term of $C_5^{\{N1'\}}$, c_2, with the cut which may lead to double-counting indicated.

FIG. 10. The third term in the sub-class $C_5^{\{N1'\}}$, c_3, with the cuts which may lead to double-counting indicated.

FIG. 11. The fourth term in $C_5^{\{N1'\}}$, c_4, and the cut which may lead to double-counting.

FIG. 12. The two possible cuts which may lead to double-counting fifth term of $C_5^{\{N1'\}}$, c_5.

FIG. 13. The two possible cuts which may lead to double-counting in the sixth term of $C_5^{\{N2'\}}$, c_6.

FIG. 14. The definition of the new vertex, $f^{(1)*}$.

FIG. 15. Two diagrams which are part of $T_{NN}^{(2)}$. They will both contribute to the dressing of the vertices in one-pion exchange when the one-pion exchange piece of $T_{NN}^{(1)}$ is inserted, as shown here. Hence both diagrams will lead to double-counting, unless the vertex used in the one-pion exchange piece of $T_{NN}^{(2)}$ is adjusted.

FIG. 16. On the right we show the three terms D_1, D_2 and D_π. On the left we depict, in each case, the two different places they appear in the pion absorption/production piece of $T^{(2)}_{NN}$.

FIG. 17. In the lower half of this diagram we show the two terms B and X, while the upper half displays the two places they appear in the pion absorption/production piece of $T^{(2)}_{NN}$.

FIG. 18. A diagram containing the input NN amplitude $t^{(1)}_{NN}$. Note that if the one-pion exchange piece of the NN amplitude is considered this diagram yields the crossed two-pion exchange graph.

FIG. 19. A piece of an arbitrary Feynman diagram to be included in the four-dimensional $NN - \pi NN$ equations.

FIG. 20. An example of a diagram in which a self-energy contribution on one of the legs belonging to the initial state may be isolated.